university of groningen
faculty of mathematics and natural sciences

Approximating Manifolds by Meshes: Asymptotic Bounds in Higher Codimension

David de Laat

Master's Thesis in Mathematics

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Summary

We discuss asymptotic upper bounds on the Hausdorff distance between manifolds and optimal meshes. Here a mesh is a geometric simplicial complex whose carrier is topologically equivalent to the manifold and whose vertices lie on the manifold. By equipping manifolds with new curvature induced metrics we generalize a method of Clarkson, which uses nets and the second fundamental form to mesh hypersurfaces, to higher codimension. This yields new upper bounds for manifolds which admit global nonoriented normal frame fields, and these bounds compare well to bounds which are already known in special cases. Our approach yields an explicit expression for the constant in the asymptotic upper bound.

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Author: David de Laat
First supervisor: Prof. dr. Gert Vegter
Second supervisor: Dr. Mirjam E. Dür
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Johann Bernoulli Institute for Mathematics and Computer Science
Groningen, the Netherlands

Contents

Chapter 1

Introduction

1.1 Meshing and Approximation Errors

Manifold meshing is the process of approximating a manifold by a finite collection of simple elements. In this report these simple elements will be linear simplices whose vertices lie on the manifold, so for instance the approximation of a closed curve by line segments or a compact surface by triangles; see Figure 1.1. This is more precisely called *piecewise linear manifold meshing*, but in the remainder of this report we will refer to this as *meshing*.

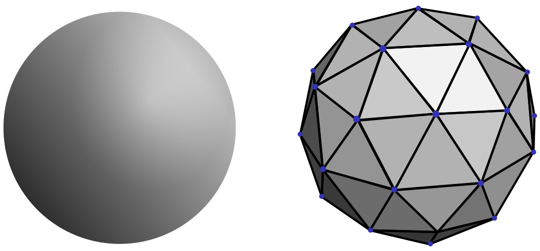

Figure 1.1: Approximation of a sphere by a mesh.

Meshing has applications in a variety of fields. In practice it is often convenient to define a surface implicitly, that is, as the zero set of smooth functions. This holds for well known shapes such as spheres and ovaloids, but by using a function like

$$f(x) = \sum_{i=1}^{n} \exp(-\|A_i x - b_i\|) - n,$$

where the A_i are matrices and the b_i vectors, this also holds for more flexible shapes [2, Page 182]. However, in order to visualize a surface, to perform efficient collision detection, or to do other computations at surfaces or the regions inside and outside surfaces, it is often easier to work with a collection of triangles. Meshing therefore has applications in for instance computer graphics, robotics, and engineering. The above example of implicitly defined surfaces can easily be extended to higher codimension by increasing the number of functions and applications for meshing in higher codimension exist in the fields of for example data analysis and machine learning.

Given a compact manifold embedded in a Euclidean space, we define a *mesh* to be a geometrical simplicial complex whose carrier is homeomorphic to the manifold and whose vertices lie on the manifold. Here the requirement that the carrier is homeomorphic to the manifold is sometimes replaced by the stronger requirement that the mesh is ambiently isotopic to the manifold. We define the quality of a mesh by the Hausdorff distance between the manifold and the carrier of the mesh, and we call this the *approximation error*. Given a fixed number of n vertices, we define an *optimal mesh* to be a mesh which minimizes the approximation error. The approximation error of an optimal mesh is called the *optimal approximation error*. If n grows to infinity, then the optimal approximation error converges to zero, and the main topic of this report is the construction of bounds on the speed of this convergence.

Fejes Tóth showed in in 1947 that the optimal approximation error of a space curve is asymptotically equal to

$$\frac{1}{8n^2}\left(\int_0^l \sqrt{|\kappa(s)|}\,\mathrm{d}s\right)^2 \quad \text{as} \quad n \to \infty,$$

where $\kappa(s)$ is the curvature of the curve which is parametrized by arc length [16]. This means that if the curve is embedded in a Euclidean space of dimension higher than 2, the constant factor depends only on the curvature of the curve, and for instance not on its torsion. In this same article Fejes Tóth discusses the meshing of 2-dimensional convex hypersurfaces, which are hypersurfaces which bound strictly convex regions, that is, they have positive curvature everywhere.

In 1986, Schneider showed that the optimal approximation error of a convex d-hypersurface is asymptotically equal to

$$\frac{1}{2}\left(\frac{\lambda_d \int_M \sqrt{|K(p)|}\,\mathrm{d}\sigma(p)}{nV_d}\right)^{2/d} \quad \text{as} \quad n \to \infty,$$

where $K(p)$ is the Gaussian curvature at p [15], V_d the volume of a unit Euclidean d-ball, λ_d the packing number of Euclidean d-space, and σ is the Euclidean volume measure. Although the result of Schneider holds for manifolds of class C^3, in 1993, Gruber improved this result to manifolds of class C^2 [6].

Clarkson showed in 2006 that by replacing the Gaussian curvature by the absolute value of the Gaussian curvature, a similar result can be obtained for nonconvex hypersurfaces [3]. He showed that the optimal approximation error of an optimal mesh of a hypersurface is asymptotically upper bounded by

$$c_d\left(\frac{1}{n}\int_M \sqrt{|K(p)|}\,\mathrm{d}\sigma(p)\right)^{2/d} \quad \text{as} \quad n \to \infty.$$

In this report we will generalize this result of Clarkson to higher codimension and we will give an explicit formula for the constant in the asymptotic upper bound. This constant is comparable to the constant given by Schneider and Gruber for convex hypersurfaces.

1.2 Structure of the Report

As discussed in the previous section, we give a generalization of a method of Clarkson to obtain asymptotic upper bounds on the optimal approximations error of manifolds by meshes. The description of this method is spread out over multiple chapters, and we will use this section to give an overview of the contents of these chapters.

In Chapter 2 we discuss some preliminaries. In particular, we discuss the concept of Hausdorff distance which is used to define the approximation error of a mesh. We also discuss asymptotic inequalities, which can be regarded as the one sided equivalent of asymptotic equalities. The notation of asymptotic inequalities is used to give asymptotic upper bounds on the approximation error.

An important technique in this report is the use of ϵ-nets, as they are used to distribute the vertices of a mesh well space with respect to a metric of the space. We discuss their relevant properties in Chapter 3. An ϵ-net is defined to be both an ϵ-covering and an ϵ-packing, and we use the latter property to bound ϵ in the cardinality of the net for spaces where the metric satisfies a certain regularity condition. We show that for Riemannian manifolds a packing result of Gruber [6] gives an optimal bound for ϵ in the number of points.

In Chapter 4 we define so called *curvature metrics* on embedded manifolds such that distances are large in regions of high curvature, and vice versa. To do this we show that for each point p on the manifold there is a neighbourhood of p which is the graph of c height functions, where c is the codimension. The directions of these height functions are determined by an orthonormal frame of the normal space at p. We define a *curvature matrix* at p to be the sum of the convexified Hessian matrices of these height functions, where by convexified we mean that we take the absolute value of the eigenvalues. Due to this convexification the curvature matrices are invariant under sign changes of the vectors in the orthonormal normal frame. In codimension 1, the determinant of such a curvature matrix is the absolute value of the Gaussian curvature at p.

By using a nonoriented orthonormal normal frame field we obtain a set of curvature matrices for each point of the manifold. We use this to define a Riemannian tensor field called a *curvature tensor field*. Such a field induces the desired *curvature metric*. Such fields also induce measures which we will use in the constants of the upper bounds. In codimension one this measure is equal to the integral of the square root of the absolute value of the Gaussian curvature. In Chapter 7 we discuss the relation between these curvature tensor fields, second fundamentals forms, and shape operators.

We will use a net in a curvature metric as the vertex set of an approximating mesh. In this way there will be relatively many vertices in regions of high curvature, which will be useful in bounding the approximation errors. In Chapter 5 we discuss the construction of meshes. We use a result of Leibon and Letscher [11] to show that for any sufficiently small $\epsilon > 0$, there exist Delaunay triangulations with an ϵ-net as vertex set. Here a *triangulation* is a simplicial complex together with a homeomorphism from the carrier of this complex to the manifold, and a *Delaunay triangulation* is a triangulation where for each simplex there is a geodesic disk ball such that the vertices of the simplex lie on the boundary of this ball and the interior of the ball does not contain any vertices. We show that for ϵ sufficiently small, the linear approximation of such a triangulation is a mesh, where by *linear approximation* we mean that we replace each embedded simplex by a linear simplex with the same vertices. A mesh which is also a Delaunay triangulation is called an *intrinsic Delaunay mesh*.

In Chapter 6 we discuss upper bounds on intrinsic Delaunay meshes which have sufficiently dense vertex sets in a curvature metric. We use that the vertex set is an ϵ-packing to bound ϵ in the number of vertices. Then we use that the vertex set is an ϵ-covering and that the mesh is in particular a Delaunay triangulation, to bound the approximation error in ϵ. For manifolds which admit nonoriented orthonormal normal frame fields, this yields asymptotic upper bounds on the optimal approximation errors in the number of vertices. In Section 6.2 we discuss special cases of these results which allows us to compare the bounds with already known results. In Section 6.3.2 we discuss, as an example, upper bounds on the optimal approximation errors for flat embeddings of the 2-dimensional torus in 4-dimensional space.

In Chapter 5 there are some results which we only stated for the case where the manifolds are one or two dimensional, since we do currently lack proofs for the more general case. Due to these restrictions the main theorem in Chapter 6 also only holds for small dimensions (but for arbitrary codimension). We discuss these open problems in Chapter 8. In this chapter we also discuss the generalization of these bounds to manifolds with boundary, which could aid in the removal of the condition that the manifold must admit a nonoriented orthonormal normal frame field.

1.3 Acknowledgements

I would like to express my gratitude to my supervisor, Gert Vegter, for his guidance and useful comments throughout the process of writing this thesis. I would also like to thank my second supervisor, Mirjam Dür, for appraising this thesis and Mathijs Wintraecken for reading drafts of this report and for his many helpful suggestions.

Chapter 2

Preliminaries

In this chapter we introduce some notation which will be used in the rest of the text. Sections 2.1 and 2.2 serve mainly to introduce notation which is new or not widely used, but wich will be useful in later sections. In Section 2.3 we discuss the important and widely used concept of Hausdorff distance. In Section 2.4 we discuss Riemannian manifolds and Riemannian structure induced metrics and measures.

2.1 Asymptotic Inequalities

In this section we will introduce asymptotic notation which can be regarded as the one sided version of asymptotic equality. In later sections we will use this notation to state asymptotic upper and lower bounds.

Two functions $f(n)$ and $g(n)$ are *asymptotically equal* if $\lim_{n\to\infty} f(n)/g(n) = 1$. Asymptotic equality is an equivalence relation, so we denote it by

$$f(n) \sim g(n) \quad \text{as} \quad n \to \infty.$$

If for each $C > 1$ there is an $N \geq 0$ such that $f(n) \leq Cg(n)$ for all $n \geq N$, then we use the notation

$$f(n) \lesssim g(n) \quad \text{as} \quad n \to \infty.$$

This notion satisfies reflexivity and transitivity, and as the following trivial lemma shows, it also satisfies antisymmetry. This final property is an argument to use this definition as the *one sided* version of asymptotic equality.

Lemma 2.1. *If $f(n)$ and $g(n)$ are functions, then*

$$f(n) \sim g(n) \quad \textit{as} \quad n \to \infty,$$

if and only if

$$f(n) \lesssim g(n) \quad \textit{and} \quad g(n) \lesssim f(n) \quad \textit{as} \quad n \to \infty.$$

The above notions can be extended in the obvious way to limits of the form $x \to c$, $x \downarrow c$, and $x \uparrow c$, where c is some constant. The notation extends to multiple variables in the following way: If for each $C > 1$ there are $N, M \geq 0$ such that $f(n,m) \leq Cg(n,m)$ for all pairs (n,m) where $n \geq N$ and $m \geq M$, then we use the notation

$$f(n,m) \lesssim g(n,m) \quad \text{as} \quad n \to \infty \quad \text{and} \quad m \to \infty.$$

2.2 Operations on Diagonalizable Matrices

Given a function $\phi : \mathbb{R} \to \mathbb{R}$ and a diagonal matrix D, we denote by $\phi(D)$ the entrywise application of ϕ to the diagonal elements of D. We could generalize this to arbitrary matrices by defining $\phi(A)$ to be the entrywise application of ϕ to the elements of A, but the following definition, which works only for diagonalizable matrices, turns out to be more useful.

For diagonalizable A, let $\phi(A) = X\phi(D)X^{-1}$, where $A = XDX^{-1}$ is some diagonalization. In the following lemma we show that this definition is independent of this diagonalization. Note that for linear ϕ, this definition yields the same result as the entrywise definition.

Lemma 2.2. *If $\phi : \mathbb{R} \to \mathbb{R}$ is a function and A is a diagonalizable matrix with diagonalizations XDX^{-1} and $\tilde{X}\tilde{D}\tilde{X}^{-1}$, then*

$$X\phi(D)X^{-1} = \tilde{X}\phi(\tilde{D})\tilde{X}^{-1},$$

where $\phi(D)$ denotes the entrywise application of ϕ to the diagonal elements of D.

Proof. Since $XDX^{-1} = \tilde{X}\tilde{D}\tilde{X}^{-1}$ we have $DZ = Z\tilde{D}$, where $Z = X^{-1}\tilde{X}$. That is,

$$d_{ii}z_{ij} = \tilde{d}_{jj}z_{ij} \quad \text{for all} \quad i, j.$$

By considering the cases $z_{ij} = 0$ and $z_{ij} \neq 0$ separately, we obtain

$$\phi(d_{ii})z_{ij} = \phi(\tilde{d}_{jj})z_{ij} \quad \text{for all} \quad i, j,$$

so $\phi(D)Z = Z\phi(\tilde{D})$, which completes the proof. □

We can always use this notation for symmetric matrices, since they are diagonalizable by the spectral theorem. The special case where $\phi(x) = |x|$ will be used to obtain a positive semidefinite matrix from a symmetric matrix. The case where $\phi(x) = \sqrt{x}$ will be used to take the square root of positive semidefinite matrices.

2.3 Hausdorff Distance

When we approximate manifolds embedded in Euclidean spaces by polytopes, we will define the approximation error as the Hausdorff distance between the manifold and the polytope. In this section we will give the definition of Hausdorff distance and we will give some basic definitions which we will use when working with metric spaces.

Let (M, d) be a metric space. The *(geodesic) closed ball* and *(geodesic) sphere* of radius R centred around x are given by

$$B_{\mathrm{d}}(x; R) = \{y \in M \mid \mathrm{d}(x, y) \leq R\}$$

and

$$S_{\mathrm{d}}(x; R) = \{y \in M \mid \mathrm{d}(x, y) = R\}.$$

For $x \in M$ and $A, B \subset M$ let

$$\mathrm{d}(x, B) = \inf_{b \in B} \mathrm{d}(x, b) \quad \text{and} \quad \mathrm{d}(A, B) = \sup_{a \in A} \mathrm{d}(a, B).$$

The *Hausdorff distance* between the sets $A, B \subset M$ is given by

$$\mathrm{d}_H(A, B) = \max\{\mathrm{d}(A, B), \mathrm{d}(B, A)\}.$$

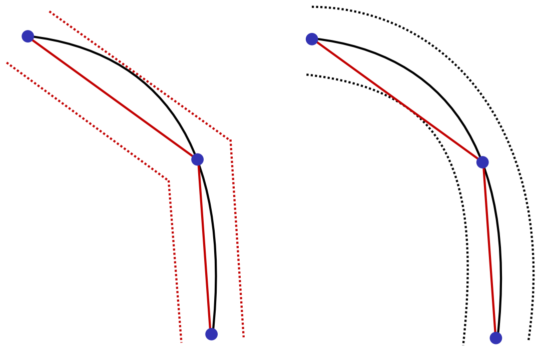

Figure 2.1: Smallest tubular neighbourhoods.

We will use tubular neighbourhoods to give an alternative, more geometric, definition of Hausdorff distance. The *tubular neighbourhood* of a subset $S \subset M$, with radius R, is given by

$$\{x \in M \mid \mathrm{d}(x, S) \leq R\}.$$

The Hausdorff distance between sets A and B is the smallest number $R \geq 0$ such that the tubular neighbourhood of radius R around A contains B, and such that the tubular neighbourhood of radius R around B contains A.

2.4 Riemannian Metric Measure Spaces

We will use the Riemannian structure q of a manifold M to define a metric d and a measure μ on M. This yields a metric measure space (M, d, μ), which will be called a *Riemannian metric measure space*. Although it is quite customary to only discuss C^∞ manifolds and structures, here we will be more precise. One reason for this is that in many places in this report the tensor fields are only required to be of class C^0.

Given a manifold M of class C^k with $k \geq 1$, we can add additional structure to the manifold in the form of a Riemannian tensor field. A *Riemannian tensor field* of class C^k on M is a positive definite covariant 2-tensor field of class C^k on M, and is also called a *Riemannian metric*. A *covariant 2-tensor field* q of class C^k on M is a bilinear form q_p on each tangent space T_pM, such that the map

$$M \to \mathbb{R},\ p \mapsto q_p(u_p, v_p)$$

is of class C^k for any two C^k vector fields u and v on M. The form q is *symmetric*if $q_p(u_p, v_p) = q_p(v_p, u_p)$ for all vector fields u and v on M and all $p \in M$. The form q is *positive definite*if it is symmetric and if $q_p(u_p, v_p) > 0$ for all $p \in M$ and all nonzero $u_p, v_p \in T_pM$.

A *Riemannian d-manifold* is a d-manifold of class C^k together with a Riemannian tensor field of class C^l, with $k \geq 1$ and $k \geq l \geq 0$. When a Riemannian tensor field q is fixed on a manifold M, we will refer to it as the *Riemannian structure* of M.

Metric. Let q be a Riemannian tensor field of class C^k on M. Denote by $\Gamma^k(p, q)$ the set of piecewise C^k curves $\gamma : [0, 1] \to M$, with $\gamma(0) = p$ and $\gamma(1) = q$. Let

$$\mathrm{d}(p, q) = \inf_{\gamma \in \Gamma^k(p,q)} \int_0^1 \sqrt{q_{\gamma(t)}(\gamma'(t), \gamma'(t))}\, \mathrm{d}t,$$

for all $p, q \in M$. This defines a metric whose topology coincides with the topology of M [8, Lemma and Corollary 1.4.1]. We call d the metric induced by the Riemannian structure, and since it is induced by a C^k Riemannian tensor field, we call it a C^k metric.

When a metric space is also a d-manifold, as is the case with the just constructed metrics, then we will only use the notation $B_{\mathrm{d}}(x; R)$, as defined in Section 3.1, if x and R are such that $B_{\mathrm{d}}(x; R)$ is homeomorphic to a closed Euclidean d-ball.

Measure. Let $\{x_\alpha : U_\alpha \to \mathbb{R}^d \mid \alpha \in A\}$ be an atlas of M and $\{\phi_\alpha \mid \alpha \in A\}$ a subordinate partition of unity. Let

$$\omega_p = \sum_{\alpha \in A} \phi_\alpha(p) \sqrt{g_\alpha(p)}\, \mathrm{d}x_\alpha^1 \wedge \cdots \wedge \mathrm{d}x_\alpha^d,$$

where $g_\alpha(p)$ is the determinant of the matrix

$$g_\alpha^{ij}(p) = q_p \left(\frac{\partial}{\partial x_\alpha^i}\Big|_p, \frac{\partial}{\partial x_\alpha^j}\Big|_p \right).$$

Let $\mu(V) = \int_V \omega$ for Borel sets $V \subset M$. This defines a measure, which we call the measure induced by the Riemannian structure.

Hausdorff measure. The Riemannian structure induced measure is the *Hausdorff measure* with respect to the Riemannian structure induced metric, that is,

$$\mu(V) = V_d \lim_{\delta \downarrow 0} \inf_{C \in \mathcal{C}_\delta} \sum_{S \in C} \mathrm{diam}_{\mathrm{d}}(S)^d.$$

Here the *diameter* of a set $S \subset M$ is defined as $\mathrm{diam}_{\mathrm{d}}(S) = \sup_{x,y \in S} \mathrm{d}(x, y)$, and $\mathcal{C}_\delta$ is the collection of finite or countable coverings of $V \subset M$ by sets with diameter less than δ. The volume of a Euclidean unit d-ball is given by

$$V_d = \frac{\pi^{d/2}}{\Gamma(1 + d/2)},$$

where Γ denotes the Gamma function [7]. Note that $V_0 = 1$ and $V_d \to 0$ as $d \to \infty$.

Chapter 3

Nets

In this section we define nets which are discrete sets in a metric space whose points are nicely distributed with respect to the metric. In Section 5 we will use these nets as vertex sets of polytopal approximations of manifolds. This nice distribution allows us to asymptotically bound the cardinality of an ϵ-net in ϵ. For a simple proof of such bounds we will use the concept of dimension regular metric measure spaces. We show that the Riemannian structure of a manifold induces a dimension regular metric measure space. Hereby we show that the bounds obtained for dimension regular spaces hold in particular for Riemannian manifolds. In the final section of this chapter we use a result of Gruber to show that better bounds can be obtained by directly using the Riemannian structure of the manifold.

3.1 Definitions and Properties

In this section we define ϵ-nets and discuss a few important properties of these nets in metric spaces and more specifically in metric manifolds.

An *ϵ-covering* of (M, d) is a set $S \subset M$ such that $\mathrm{d}(x, S) < \epsilon$ for all $x \in M$. That is, the collection $\{\mathrm{int}(B_{\mathrm{d}}(x; \epsilon)) \mid x \in S\}$ is an open cover of M. An *ϵ-packing* of (M, d) is a set $S \subset M$ such that $\mathrm{d}(x, S \setminus \{x\}) > \epsilon$ for all $x \in S$. That is, the $\epsilon/2$-balls around the points in S are disjoint. An *ϵ-net* of (M, d) is a set $S \subset M$ that is both an ϵ-covering and an ϵ-packing.

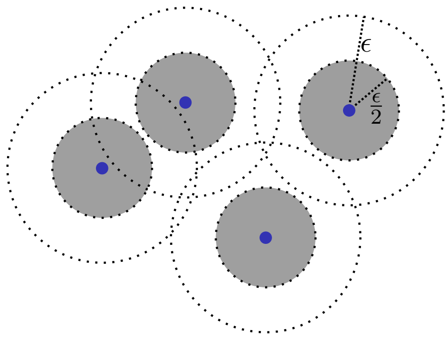

Figure 3.1: Part of an ϵ-net in the plane.

The following lemma shows, in particular, that ϵ-nets of compact spaces are finite. We will use this simple result to show that ϵ-nets always exist.

Lemma 3.1. *If* (M, d) *is a compact metric space and* $\epsilon > 0$*, then every* ϵ*-packing* S *of* (M, d) *is finite.*

Proof. The open cover

$$C = \{\mathrm{int}(B_{\mathrm{d}}(x; \epsilon)) \mid x \in S\} \cup (M \setminus \{\mathrm{int}(B_{\mathrm{d}}(x; \epsilon/2)) \mid x \in S\})$$

contains a finite subcover, since M is compact. None of the sets in C can be removed, however. So C must be finite already, which implies that S is finite. □

From the depiction in Figure 6.1 of a section of an ϵ-net in the Euclidean plane we see that ϵ-nets in the plane are easily constructed. The following lemma shows that this also holds for any compact metric space.

Lemma 3.2. *If* (M, d) *is a compact metric space and* $\epsilon > 0$*, then there exists a finite* ϵ*-net of* (M, d).

Proof. Let $(x_n)_{n=1}^{\infty}$ be a sequence such that

$$x_{n+1} = \mathrm{argmax}_{x \in M} \mathrm{d}(x, S_n)$$

for each n, with $S_n = \{x_1, \ldots, x_n\} \subset M$. Such a sequence can be constructed by a greedy algorithm and exists since M is compact.

There exists a smallest finite number $N \geq 1$ such that $\mathrm{d}(x_{N+1}, S_N) \leq \epsilon$. For otherwise,

$$\mathrm{d}(x, S_n \setminus \{x\}) \geq \mathrm{d}(x_n, S_{n-1}) > \epsilon$$

for all $x \in S_n$ and all $n \geq 1$. This would yield a countable ϵ-packing $\cup_{n=1}^{\infty} S_n$, which contradicts the result of Lemma 3.1. The set S_N is an ϵ-packing since N is the smallest number for which $\mathrm{d}(x_{N+1}, S_N) \leq \epsilon$, that is, $\mathrm{d}(x, S_N \setminus \{x\}) > \epsilon$ for all $x \in S_N$. The set S_N is an ϵ-covering, since $\mathrm{d}(x_{N+1}, S_N) \leq \epsilon$ implies $\mathrm{d}(x, S_N) \leq \epsilon$ for all $x \in M$. □

A *metric d-manifold* M is a manifold with a metric d defined on it such that the topology induced by the metric agrees with the manifold topology. A point set $S \subset M$ is said to be in *general position* if no $d + 2$ points of S lie on the boundary of some ball $B_{\mathrm{d}}(x; R)$. We have seen that ϵ-nets always exist in compact metric spaces; the following lemma shows that if a metric space is equipped with a fitting manifold structure, then ϵ-nets which are in general position always exist.

Lemma 3.3. *If* (M, d) *is a compact metric d-manifold and* $\epsilon > 0$*, then there exists a finite* ϵ*-net of* (M, d) *which is in general position.*

Proof. By Lemma 3.2, there exists a finite ϵ-net $\{x_1, \ldots, x_n\}$ of (M, d). Let y_i be a point near x_i such that no more than $d + 1$ points of $\{y_1, \ldots, y_i\}$ lie on a geodesic sphere. Let C_i be the union of geodesic spheres that meet at least $d + 1$ points of the set $\{y_1, \ldots, y_i\}$. The points y_i can be chosen arbitrarily close to x_i since $B_{\mathrm{d}}(y_i; R) \setminus (C_i \cap B_{\mathrm{d}}(y_i; R))$ is nonempty for all $1 \leq i \leq n$ and all $\mathbb{R} > 0$.

The collection of ϵ-balls around the points $\{x_1, \ldots, x_n\}$ is an open covering of M, so by taking the points y_i close enough to x_i, the set of ϵ-balls around the points $\{y_1, \ldots, y_n\}$ also is an open covering of M. Hence $\{y_1, \ldots, y_i\}$ is an ϵ-covering.

Similarly, the closed $\epsilon/2$-balls around the points $\{x_1, \ldots, x_n\}$ are disjoint, so by taking the points y_i close enough to x_i, the $\epsilon/2$ balls around the points $\{y_1, \ldots, y_n\}$ are also disjoint, so $\{y_1, \ldots, y_n\}$ is an ϵ-packing. □

The following lemma shows that for topological spaces with more than one metric defined on it, for each $\delta > 0$, an ϵ-covering in one metric is a δ-covering in another metric by taking ϵ sufficiently small.

Lemma 3.4. *Let (X, d_1) and (X, d_2) be metric spaces such that d_1 and d_2 induce the same compact topology on X and let $\delta > 0$. For all $\epsilon > 0$ sufficiently small, each ϵ-covering in (X, d_1) is a δ-covering in (X, d_2).*

Proof. Since the collection $\{B_{\mathrm{d}_2}(x; \delta) \mid x \in X)$ is an open cover of the compact space X, there exists a finite set $\{x_1, \ldots, x_n\} \subset X$ such that $\{B_{\mathrm{d}_2}(x_i; \delta) \mid 1 \le i \le n\}$ is an open cover of X. Since the open sets in this cover overlap, there exists a $\sigma > 0$ such that

$$\{B_{\mathrm{d}_2}(y_i; \delta) \mid 1 \le i \le n\}$$

is an open cover of X for each choice of $y_i \in B_{\mathrm{d}_2}(x_i; \sigma)$ for $i = 1, \ldots, n$.

For each $1 \le i \le n$ there exists an $\epsilon_i > 0$ such that $B_{\mathrm{d}_1}(x_i; \epsilon_i) \subset B_{\mathrm{d}_2}(x_i; \sigma)$. Let

$$0 < \epsilon \le \min_{1 \le i \le n} \epsilon_i.$$

If N_ϵ is an ϵ-net in (X, d_1), by the above argument, N_ϵ has a point in each ball $B_{\mathrm{d}_2}(x_i; \sigma)$, hence N_ϵ is a δ-net in (X, d_2). □

3.2 Bounds using Dimension Regularity

We will give the relation, in dimension regular metric measure spaces, between ϵ and the cardinality of an ϵ-net as $\epsilon > 0$ converges to 0. This will allow us to give an asymptotic bound for the number of points of an ϵ-net as a function of ϵ.

A metric measure space (M, d, μ) is called *dimension regular* with *dimension d* if there exists numbers $\lambda > 1$ and $\mathcal{E} > 0$ such that

$$\frac{1}{\lambda}\epsilon^d \le \mu(B_{\mathrm{d}}(x; \epsilon)) \le \lambda \epsilon^d$$

for all $x \in M$ and all $0 < \epsilon < \mathcal{E}$. A metric measure space (M, d, μ) is called *strongly dimension regular* with dimension d if there exists an $\mathcal{E}_\lambda > 0$ for all $\lambda > 1$ such that

$$\frac{1}{\lambda} V_d \epsilon^d \le \mu(B_{\mathrm{d}}(x; \epsilon)) \le \lambda V_d \epsilon^d$$

for all $x \in M$ and all $0 < \epsilon < \mathcal{E}_\lambda$. We emphasize that in the above definitions the numbers $\mathcal{E}$ and $\mathcal{E}_\lambda$ do not depend on the point $x \in M$.

Lemma 3.5. *If the compact metric measure space (M, d, μ) is dimension regular with dimension d, then for each $\epsilon > 0$ there exists a finite ϵ-net S_ϵ with cardinality N_ϵ such that*

$$N_\epsilon = \Theta(\epsilon^{-d}) \quad as \quad \epsilon \downarrow 0.$$

If M is strongly dimension regular with dimension d, then additionally

$$\frac{\mu(M)}{V_d \epsilon^d} \lesssim N_\epsilon \lesssim 2^d \frac{\mu(M)}{V_d \epsilon^d} \quad as \quad \epsilon \downarrow 0.$$

Proof. By Lemma 3.2 there exists an ϵ-net S_ϵ with finite cardinality N_ϵ for each $\epsilon > 0$, hence we can write $S_\epsilon = \{x_1, \ldots, x_{N_\epsilon}\}$. By the definition of an ϵ-net we have

$$\sum_{n=1}^{N_\epsilon} \mu(B_{\mathrm{d}}(x_n; \epsilon/2)) \leq \mu(M) \leq \sum_{n=1}^{N_\epsilon} \mu(B_{\mathrm{d}}(x_n; \epsilon)). \tag{3.1}$$

For the case where the metric measure space is dimension regular, it follows from (3.1) that there are numbers $d \in \mathbb{N}$ and $\lambda > 1$ such that

$$\frac{N_\epsilon \epsilon^d}{\lambda 2^d} \leq \mu(M) \leq \lambda N_\epsilon \epsilon^d,$$

for $\epsilon > 0$ sufficiently small. So

$$N_\epsilon = \Theta(\epsilon^{-d}) \quad \text{as} \quad \epsilon \downarrow 0.$$

If (M, d, μ) is strongly dimension regular then it follows from (3.1) that there is a $d \in \mathbb{N}$ such that for all $\lambda > 1$,

$$\frac{N_\epsilon V_d \epsilon^d}{\lambda 2^d} \leq \mu(M) \leq \lambda N_\epsilon V_d \epsilon^d,$$

for all $\epsilon > 0$ sufficiently small, so

$$\frac{\mu(M)}{V_d \epsilon^d} \lesssim N_\epsilon \lesssim 2^d \frac{\mu(M)}{V_d \epsilon^d} \quad \text{as} \quad \epsilon \downarrow 0. \qquad \square$$

In the above lemma we have the requirement that the metric measure space is dimension regular. The following simple example shows that this result does not hold for arbitrary compact metric measure spaces. Let S be a finite set with cardinality n equipped with the discrete topology. Here $N_\epsilon = n$ for $\epsilon < 1$ and $N_\epsilon = 1$ for $\epsilon \geq 1$. So this means that

$$N_\epsilon \neq \Theta(\epsilon^{-d}) \quad \text{as} \quad \epsilon \downarrow 0.$$

3.3 Dimension Regularity of Riemannian Manifolds

We will show that the induced metric measure space structure of a Riemannian manifold is strongly dimension regular. This shows that we can use the bounds of Section 3.2 for all Riemannian manifolds. However, in Section 3.4 we will see that by directly using the Riemannian structure, better bounds can be obtained.

Lemma 3.6. *If M is a Riemannian d-manifold with metric* d *and measure μ induced by the Riemannian structure, then*

$$\mu(B_{\mathrm{d}}(x; \epsilon)) \sim V_d \epsilon^d \quad \text{as} \quad \epsilon \downarrow 0$$

for all $x \in M$.

Proof. Let $x \in M$ and identify $T_x M$ with $\mathbb{R}^d$ by choosing an orthonormal basis of $T_x M$. The normal neighbourhood lemma states that the exponential map $\exp_x : T_x M \to M$ restricted to $B_{\mathrm{d}_E}(0; \epsilon)$ is a diffeomorphism for $\epsilon > 0$ small enough [9, Lemma 5.10]. Since $\exp_x$ maps geodesics going through 0 to geodesics going through x, the restricted exponential map is a diffeomorphism from $B_{\mathrm{d}_E}(0;\epsilon)$ to $B_{\mathrm{d}}(x;\epsilon)$ [9, Proposition 5.11]. The Jacobian of this map is a smooth map $J : B_{\mathrm{d}_E}(0;\epsilon) \to \mathbb{R}$ with $J(0) = 1$, since $D\exp_x(0) = \mathrm{Id}$. Then

$$\begin{aligned}\mu(B_{\mathrm{d}}(x;\epsilon)) &= \int_{B_{\mathrm{d}}(x;\epsilon)} \mathrm{d}\mu(p) \\ &= \int_{B_{\mathrm{d}_E}(0;\epsilon)} (\exp_x)^* \,\mathrm{d}\mu(p) \\ &= \int_{B_{\mathrm{d}_E}(0;\epsilon)} J(p)\,\mathrm{d}\sigma(p) \\ &\sim J(0)\int_{B_{\mathrm{d}_E}(0;\epsilon)} \mathrm{d}\sigma(p) \\ &= V_d\epsilon^d,\end{aligned}$$

as $\epsilon \downarrow 0$. ☐

Lemma 3.7. *If M is a compact Riemannian d-manifold with metric* d *and measure μ induced by the Riemannian structure, then the metric measure space (M, d, μ) is strongly dimension regular with dimension d.*

Proof. Lemma 3.6 shows that

$$\epsilon_\lambda(x) = \sup\{\epsilon \in (0,1] \mid \frac{1}{\lambda}V_d\epsilon^d \le \mu(B_{\mathrm{d}}(x;\epsilon)) \le \lambda V_d\epsilon^d\}$$

is defined for all $\lambda > 1$ and all $x \in M$. The functions ϵ_λ are continuous since $\mu(B_{\mathrm{d}}(x;\epsilon))$ is continuous in both x and ϵ. In addition we have that the manifold M is compact, so

$$\mathcal{E}_\lambda = \min_{x\in M} \epsilon_\lambda(x) > 0$$

exists by the Weierstrass extreme value theorem. So, for each $\lambda > 1$ we have that

$$\frac{1}{\lambda}V_d\epsilon^d \le \mu(B_{\mathrm{d}}(x;\epsilon)) \le \lambda V_d\epsilon^d$$

for all $x \in M$ and all $0 < \epsilon < \mathcal{E}_\lambda$, which completes the proof. ☐

3.4 Bounds using the Sphere Covering and Sphere Packing Densities

In this section the structure of a Riemannian manifold is used to obtain optimal asymptotic bounds on the cardinality N_ϵ of ϵ-nets.

We define the density of a collection S of subsets of $\mathbb{R}^d$ as

$$\rho(S) = \lim_{R\to\infty} \frac{\int_{B_{\mathrm{d}_E}(0;R)} f_S(p)\,\mathrm{d}\sigma(p)}{\int_{B_{\mathrm{d}_E}(0;R)} \mathrm{d}\sigma(p)},$$

where $f_S : \mathbb{R}^d \to \mathbb{Z}_{\ge 0}$ is the map that sends a point $x \in \mathbb{R}^d$ to the number of sets in S that contain this point.

The *minimal sphere covering density* of $\mathbb{R}^d$ is defined as

$$\theta_d = \inf_{C \in \mathcal{C}_d} \rho(C),$$

where $\mathcal{C}_d$ is the collection of covers of $\mathbb{R}^d$ by Euclidean unit balls. Similarly, the *maximal sphere packing density* of $\mathbb{R}^d$ is defined as

$$\lambda_d = \sup_{P \in \mathcal{P}_d} \rho(P),$$

where $\mathcal{P}_d$ is the collection of packings of $\mathbb{R}^d$ by Euclidean unit balls. We have $\theta_d \geq 1$ and $\lambda_d \leq 1$ for all $d \geq 1$ and these bounds are obtained when $d = 1$. We can use these densities to state the following two lemmas.

Lemma 3.8 ([6, Lemma 1])**.** *If M is a compact d-manifold of class C^2 endowed with a Riemannian metric tensor of class C^0 with induced metric* d *and induced measure μ, and N_R is the minimal number of geodesic discs of radius $R > 0$ covering M, then*

$$N_R \sim \theta_d \frac{\mu(M)}{V_d R^d} \quad \textit{as} \quad R \downarrow 0.$$

Lemma 3.9 ([6, Remark 9])**.** *If M is a compact d-manifold of class C^2 endowed with a Riemannian metric tensor of class C^0 with induced metric* d *and induced measure μ, and N_R is the maximal number of disjoint geodesic discs of radius $R > 0$ in M, then*

$$N_R \sim \lambda_d \frac{\mu(M)}{V_d R^d} \quad \textit{as} \quad R \downarrow 0.$$

When we use that an ϵ-net is both an ϵ-covering and an ϵ-packing, we obtain the following corollary. The number 2^d occurs in this lemma because an ϵ-packing is defined using disjoint $\epsilon/2$-balls. Note that this corollary implies that $\theta_d \leq 2^d \lambda_d$ for all $d \geq 1$.

Corollary 3.10. *If M is a compact d-manifold of class C^2 endowed with a Riemannian metric tensor of class C^0 with induced metric* d *and induced measure μ, and for each $\epsilon > 0$ we have an ϵ-net S_ϵ in (M, d) with cardinality N_ϵ, then*

$$\theta_d \frac{\mu(M)}{V_d \epsilon^d} \lesssim N_\epsilon \lesssim 2^d \lambda_d \frac{\mu(M)}{V_d \epsilon^d} \quad \textit{as} \quad \epsilon \downarrow 0.$$

Chapter 4

Curvature in Higher Codimension

We will use the curvature of a manifold embedded in a Euclidean space to define Riemannian tensor fields on the manifold. Such *curvature tensor fields*, as we will call them, induce *curvature metrics* on the manifold. In these metrics distances are large in regions of high curvature, and vice versa. By taking a net in such a metric the vertex density will be relatively high in regions of high curvature, which makes such a net is a suitable candidate for the vertex set of an approximating mesh.

To construct curvature tensor fields we will use the Hessian matrices of a set of height functions of which the manifold is locally the graph. This construction of curvature tensor fields for d-manifolds embedded in $\mathbb{R}^{d+c}$ is a generalization to higher codimension of the construction of a convexified second fundamental forms as done in [3].

4.1 Orthonormal Frame Fields

A compact hypersurface has an essentially unique unit normal section, but this is not the case for embedded manifolds of higher codimension. The curvature metric which will be introduced in this chapter depends on the choice of a generally nonunique normal frame. In this section we will introduce these frames and discuss important properties such as global existence.

A *frame* η of a vector space V is an ordered basis of V. A *frame field* η of class C^k of a vector bundle $\pi : E \to B$ is a frame η_p for each fibre $\pi^{-1}(\{p\})$, $p \in B$, such that the frame vectors $\eta^1, \ldots, \eta^n$ are C^k sections of the vector bundle. A frame field which is defined in an open neighbourhood of a point $p \in B$ is called a *local frame field*.

If a vector space is equipped with an inner product, then we have the concept of an *orthonormal frame*, which is a frame consisting of orthogonal unit vectors. Similarly, if a vector bundle is equipped with a positive definite bundle metric then we have the concept of an *orthonormal frame field*. The fibre bundle consisting of orthonormal frames of a vector bundle $\pi : E \to B$ is denoted by $F_o(E)$.

Let M be a smooth d-manifold embedded in $\mathbb{R}^{d+c}$. We will mostly be concerned with orthonormal frame fields of the tangent and normal bundles of M, which we call *orthonormal tangent (normal) frame fields*. Using this notation we have that a (local) orthonormal tangent frame field of M is a (local) section of $F_o(TM)$ and a (local) orthonormal normal frame field is a (local) section of $F_o(NM)$. Here NM denotes the normal bundle of M.

The tangent and normal spaces of an embedded manifold will be considered to be linear subspaces of the ambient space $\mathbb{R}^{d+c}$, where the inner product on these subspaces is inherited by the Euclidean inner product on $\mathbb{R}^{d+c}$. This means that the frame vectors of an orthonormal tangent/normal frame field form an orthogonal set of vectors in the embedded sphere $\mathbb{S}^{d+c-1}$.

Tangent frame fields. A manifold M is called *parallelizable* if its tangent bundle TM is trivial. This is the case if and only if there exists a global section in $F_o(TM)$, that is, there exists a global orthonormal tangent frame field on M.

The existence of a global orthonormal tangent frame field on a manifold M implies that there exist a nowhere zero vector field on M. Since the hairy ball theorem shows that the compact manifold $\mathbb{S}^2$ does not admit a nowhere vanishing vector field [5], it is not true in general that compact manifolds admit global orthonormal tangent frame fields.

However, for each point $p \in M$ and each orthonormal tangent frame η_p at p, we can use a chart to prove that there exists an open neighbourhood U of p such that η_p can be extended to an orthonormal tangent frame field on U.

Normal frame fields. A a compact differentiable d-manifold M embedded in $\mathbb{R}^{d+1}$ is orientable which implies that M admits two global orthonormal normal frame fields [14]. In the more general case of a compact smooth d-manifold embedded in $\mathbb{R}^{d+c}$ there does not always exist a global orthonormal normal frame field, as is shown in the following lemma.

Lemma 4.1. *There exists a smooth embedding of the projective plane $\mathbb{RP}^2$ in $\mathbb{R}^4$, and such an embedding does not admit a global orthonormal normal frame field.*

Proof. We regard $\mathbb{RP}^2$ as $\mathbb{S}^2/\sim$, where $\sim$ is the equivalence relation on $\mathbb{S}^2$ which identifies antipodal points. The map

$$\phi : (x, y, z) \mapsto \left(\frac{1}{2}xy, \frac{1}{2}xz, \frac{1}{2}y^2 - \frac{1}{2}z^2, yz\right)$$

has injective derivative for all $(x, y, z) \in \mathbb{S}^2$, so it restricts to an immersion of $\mathbb{S}^2$ in $\mathbb{R}^4$. Since ϕ is purely quadratic, it maps antipodal points to the same point, so its restriction factors to an embedding of $\mathbb{RP}^2$ in $\mathbb{R}^4$.

We will prove the second part of the lemma by contradiction. Assume that there exists orthonormal normal vector fields ν^1 and ν^2 of this embedding. Let ω be the standard volume form on $\mathbb{R}^4$ and let σ be the 2-form on this embedding defined by $\sigma = \iota_{\nu^1}\iota_{\nu^2}\omega$, where ι denotes the contraction operator. Since ν^1 and ν^2 are orthonormal everywhere, σ is nowhere zero, contradicting the nonorientability of $\mathbb{RP}^2$. □

Just as in the case of tangent frame fields, we can use a chart to show that for each point p in a manifold there exists an open neighbourhood of p such that there exists a local orthonormal normal frame field on this neighbourhood.

Nonoriented normal frame fields. Due to the convexification step in the construction of curvature matrices in Section 4.3, the sign of the normal vectors will not be important in this report. Therefore we will consider nonoriented orthonormal normal fields where opposite normal vectors are identified. We will in particular show that the class of manifolds which admit a global nonoriented orthonormal normal frame field is larger than the class of matrices which admit a global orthonormal normal frame field.

Let $\rho : \mathbb{S}^{d+c-1} \to \mathbb{RP}^{d+c-1}$ be the map that identifies antipodal points. If ν_p is an orthonormal normal frame for each $p \in M$, such that for each $1 \leq j \leq c$, the map

$$p \mapsto \rho(\nu_p^j) \in \mathbb{RP}^{d+c-1}$$

is of class C^k, then ν is called a *nonoriented orthonormal normal frame field* of class C^k.

The Möbius strip is a simple example of a manifold with boundary which does not admit a global orthonormal normal frame field but does admit a global nonoriented orthonormal normal frame field. In this report, however, we only consider manifolds without boundary.

The following lemma shows that the set of compact smooth embedded manifolds which admit a global nonoriented orthonormal normal frame field is larger than the set of compact smooth embedded manifolds which admit a global orthonormal normal frame field. In the proof we will use the concept of a *transversal vector field* ν of an embedded d-manifold $M \subset \mathbb{R}^{d+c}$, which is a smooth map $\nu : M \to \mathbb{R}^{d+c}$ such that $\nu_p \notin T_pM$ for all $p \in M$, where we view T_pM as a plane through the origin.

Lemma 4.2. *There exists an embedding of $\mathbb{RP}^2$ in $\mathbb{R}^4$ which admits a global nonoriented orthonormal normal frame field.*

Proof. Let M be the embedding of $\mathbb{RP}^2$ in $\mathbb{R}^4$ as in the proof of Lemma 4.1. The derivative at $(x, y, z) \in \mathbb{S}^2$ of $\phi : \mathbb{R}^3 \to \mathbb{R}^4$ is given by

$$D\phi(x, y, z) = \begin{pmatrix} y/2 & x/2 & 0 \\ z/2 & 0 & x/2 \\ 0 & y & -z \\ 0 & z & y \end{pmatrix}.$$

So the outer normal field (x, y, z) of $\mathbb{S}^2$ is mapped to the vector field

$$\nu_{(x,y,z)} = D\phi(x, y, z)(x, y, z) = (xy, xz, y^2 - z^2, 2yz) = 2\phi(x, y, z).$$

Since $D\phi(x, y, z)$ is injective for all $(x, y, z) \in \mathbb{S}^2$, and since $D\phi(x, y, z)$ maps $T_{(x,y,z)}\mathbb{S}^2$ surjectively on $T_{\phi(x,y,z)}M$ for all $(x, y, z) \in \mathbb{S}^2$, $D\phi(x, y, z)$ maps transversal vectors of $\mathbb{S}^2 \subset \mathbb{R}^3$ to transversal vectors of $M \subset \mathbb{R}^4$. Since we also have that $\nu_{(x,y,z)} = \nu_{(-x,-y,-z)}$, the map

$$M \to \mathbb{R}^4, p \mapsto v_{q_p},$$

where q_p is some element in $\phi^{-1}(\{p\})$, is a well defined transversal vector field of $M \subset \mathbb{R}^4$.

Projecting this transversal vector field on the normal bundle yields a nowhere vanishing normal vector field of $M \subset \mathbb{R}^4$, and normalizing this vector field yields a unit normal vector field ν^1.

We obtain a line bundle by taking the orthogonal complement of $\mathbb{R}\nu_p$ in N_pM for each $p \in M$. By picking the unit vector e_1 in an arbitrary basis of each fibre of this bundle, we obtain a nonoriented vector field ν^2. A global nonoriented orthonormal normal frame field of the embedding M of $\mathbb{RP}^2$ is then given by $\nu = \{\nu^1, \nu^2\}$. $\square$

Since a vector bundle $\pi : E \to B$ is the normal bundle of the standard embedding of B in E, there exists embedded manifolds which do not even admit a nonoriented orthonormal normal bundle. Take for example the tangent bundle of an even dimensional sphere which we view as the normal bundle of the embedding of the sphere in its tangent bundle, by the Hairy ball theorem this normal bundle does not admit a nowhere vanishing section.

Adapted frames. The combination (η_p, ν_p) of an orthonormal tangent frame η_p and an orthonormal normal frame ν_p is an orthonormal frame of $T_p\mathbb{R}^{d+c}$, and is called an *adapted frame* of M at p. Similarly, the combination (η, ν) of a local orthonormal tangent frame field and a local orthonormal normal frame field is called a *local adapted frame field*.

As we have seen, both the tangent bundle and normal bundle can be an obstruction to the existence of a global adapted frame field. We have seen, however, that local adapted frame field exists around each point.

Coordinate vectors. If $b = \{v_1, \ldots, v_d\}$ is an order basis of a vector space V, then we denote the *coordinate vector* of v with respect to the basis b by $[v]_b$.

In this report we will mostly use this notation to represent tangent vectors in local coordinates. In Section 4.7 we also look at the coordinate vectors of tangent vectors in orthonormal tangent frames of nearby points.

Lemma 4.3. *Let M be a d-manifold embedded in $\mathbb{R}^{d+c}$ of class C^1 and let $p \in M$ and $v \in T_pM$. If η is a local orthonormal tangent frame field in an open neighbourhood U_p of p of class C^1, then the map*

$$U_p \to \mathbb{R}^d, \quad q \mapsto [v]_{\eta_q}$$

is of class C^1.

Proof. If $\langle .,. \rangle$ denotes the inner product of the ambient space, then

$$[v]_{\eta_q} = (\langle v, \eta_q^1 \rangle, \ldots, \langle v, \eta_q^d \rangle).$$

The proof follows from the fact that the maps $q \mapsto \eta_q^i$ are all of class C^1, and from the fact that an inner product is smooth. $\square$

Orthonormalization. Given a set of linearly independent vectors, the Gram-Schmidt process finds an orthonormal set of vectors spanning the same space. The following lemma shows that these orthonormalized vectors will be smooth if the original vectors are smooth.

Lemma 4.4. *If $\mu^1, \ldots, \mu^d$ is a set of linearly independent sections of class C^k of a vector bundle $\pi : E \to X$ which is equipped with a C^k bundle metric, then there exists a set of orthonormal sections of class C^k of the bundle $\pi : E \to X$.*

Proof. We will use the Gram-Schmidt process to orthonormalize these vectors. Let

$$\nu_p^i = \text{normalize}(\mu_p^i - \sum_{j=1}^{i-1} \langle \mu_p^j, \nu_p^j \rangle_p \nu_p^j),$$

where

$$\text{normalize} : \mathbb{R}^d \setminus \{0\} \to \mathbb{S}^{d-1} \subset \mathbb{R}^d, \quad v \mapsto v/||v||.$$

The orthonormality of $\nu_p^1, \ldots, \nu_p^c$ follows from the linear independence of $\mu_p^1, \ldots, \mu_p^d$. Since the sections $\mu_p^1, \ldots, \mu_p^d$ are C^k, the function normalize is C^k, and the bundle metric $\langle .,. \rangle_p$ is C^k, we inductively see that the sections $\nu_p^1, \ldots, \nu_p^c$ are of class C^k. $\square$

4.2 Height Functions

We will locally write a smooth embedded manifold as the graph of a set of height functions. In order to define these functions we will use Euclidean transformations. A *Euclidean transformation* e on $\mathbb{R}^{d+c}$ is a linear map on $\mathbb{R}^{d+c}$ of the form $e(x) = Ox + t$, where O is an orthogonal matrix and t is a vector. By using the identification $T_x\mathbb{R}^{d+c} = \mathbb{R}^{d+c}$, $De(x) = O$ is constant in x. We will use the notation $De = De(0)$. The group of Euclidean transformations on $\mathbb{R}^{d+c}$ is denoted by $E(d+c)$.

Given an adapted frame (η_p, ν_p) with $\eta_p = \{\eta_p^1, \dots, \eta_p^d\}$ and $\nu_p = \{\nu_p^1, \dots, \nu_p^c\}$, there is a unique Euclidean transformation $e(\eta_p, \nu_p) \in E(d+c)$ such that $e(\eta_p, \nu_p)p = 0$ and such that

$$De(\eta_p, \nu_p)\eta_p^i = e_i \quad \text{and} \quad De(\eta_p, \nu_p)\nu_p^j = e_j \quad \text{for all} \quad 0 \le i \le d,\ 0 \le j \le c.$$

Let

$$\tau_{\eta_p} = \pi \circ e(\eta_p, \nu_p),$$

where $\pi : \mathbb{R}^{d+c} \to \mathbb{R}^d$ is the map that drops the last c coordinates. Note that τ_{η_p} does not depend on the orthonormal normal frame ν_p, as suggested by the notation.

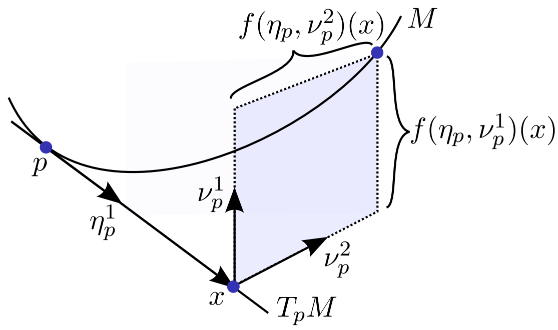

Figure 4.1: Height functions in codimension 2.

Lemma 4.5. *Let M be a d-manifold of class C^k embedded in $\mathbb{R}^{d+c}$, where $k \ge 1$. Furthermore, let $p \in M$ and let (η_p, ν_p) be an adapted frame at p. There exists an open neighbourhood U_p of p and C^k functions*

$$f(\eta_p, \nu_p^1), \dots, f(\eta_p, \nu_p^c) : \tau_{\eta_p}(U_p) \to \mathbb{R},$$

such that $\tau_{\eta_p} : U_p \to \mathbb{R}^d$ is a homeomorphism,

$$e(\eta_p, \nu_p)q = \big(\tau_{\eta_p}q, f(\eta_p, \nu_p^1)(\tau_{\eta_p}q), \dots, f(\eta_p, \nu_p^c)(\tau_{\eta_p}q)\big)$$

for all $q \in U_p$, and

$$f(\eta_p, \nu_p^i)(0) = 0 \quad \textit{and} \quad Df(\eta_p, \nu_p^i)(0) = 0 \quad \textit{for all} \quad 1 \le i \le c.$$

Proof. The inclusion map $\iota = (\iota_1, \dots, \iota_{d+c}) : e(\eta_p, \nu_p)M \to \mathbb{R}^{d+c}$ is an embedding, so it is an immersion, which means that $D\iota(0)$ is injective. Since $T_0 e(\eta_p, \nu_p)M$ is the $x_{d+1} = \ldots = x_{d+c} = 0$ plane, we have $D\iota_{d+1}(0) = \ldots = D\iota_{d+c}(0) = 0$. Let

$$\psi = (\iota_1, \dots, \iota_d) : e(\eta_p, \nu_p)M \to \mathbb{R}^{d+c},$$

the linear map $D\psi(0)$ is bijective by an application of the rank-nullity theorem.

By the inverse function theorem there exists open neighbourhoods $U_p \subset M$ of p and $V \subset \mathbb{R}^d$ of 0 such that $\psi : e(\eta_p, \nu_p)U_p \to V$ is a diffeomorphism. Let

$$f(\eta_p, \nu_p^i) = \iota_{d+i} \circ \psi^{-1},$$

for $i = 1, \ldots, c$, then

$$\big(\psi(q), f(\eta_p, \nu_p^1)(\tau_{\eta_p} q), \ldots, f(\eta_p, \nu_p^c)(\tau_{\eta_p} q)\big) = (\psi(q), \iota_{d+1}(q), \ldots, \iota_{d+c}(q)) = \iota(q) = q,$$

for all $q \in e(\eta_p, \nu_p)U_p$. Let $\tau_{\eta_p} = \psi \circ e(\eta_p, \nu_p) : U_p \to \mathbb{R}$, then

$$\begin{aligned} e(\eta_p, \nu_p)q &= \iota(e(\eta_p, \nu_p)q) \\ &= (\psi(e(\eta_p, \nu_p)q), \iota_{d+1}(e(\eta_p, \nu_p)q), \ldots, \iota_{d+c}(e(\eta_p, \nu_p)q)) \\ &= (\tau_{\eta_p}(q), f(\eta_p, \nu_p^1)(\tau_{\eta_p} q), \ldots, f(\eta_p, \nu_p^c)(\tau_{\eta_p} q)), \end{aligned}$$

for all $q \in U_p$. This completes the main portion of the lemma.

Furthermore, we have

$$f(\eta_p, \nu_p^i)(0) = \psi_{d+i}(\iota^{-1}(0)) = 0$$

and

$$\begin{aligned} Df(\eta_p, \nu_p^i)(0) &= D\iota_{d+i}(\psi^{-1}(0))D\psi^{-1}(0)) \\ &= D\iota_{d+i}(0)D\psi^{-1}(0) = 0D\psi^{-1}(0) = 0. \end{aligned}$$

□

Let U_p be the maximal open neighbourhood of p for which τ_{η_p} is injective. The neighbourhood $U_p$is well defined and depends only on the point p. From now on we restrict the domain of τ_{η_p} to U_p, such that τ_{η_p} is a chart of M. The functions

$$f(\eta_p, \nu_p^1), \ldots, f(\eta_p, \nu_p^c) : \tau_{\eta_p}(U_p) \to \mathbb{R}$$

are called the *height functions* of M corresponding to the adapted frame (η_p, ν_p). In the rest of the text we will keep using the same notation for the charts $\tau_{\eta_p} : U_p \to \mathbb{R}^d$ and the height functions as constructed above.

4.3 Curvature Matrices and Curvature Numbers

If the embedded manifold M is of differentiability class C^2, then the height functions $f(\eta_p, \nu_p^i)$ are also of class C^2. From the definition of these height functions we see that

$$f(\eta_p, \nu_p^i)(0) = 0 \quad \text{and} \quad Df(\eta_p, \nu_p^i)(0) = 0.$$

It are the second order derivatives – the coefficients of the Hessian matrices $Hf(\eta_p, \nu_p^i)(0)$ – that contain the curvature information of the manifold. These matrices are symmetric, so by using the notation of Section 2.2 we can define

$$C(\eta_p, \nu_p^i) = \left| Hf(\eta_p, \nu_p^i)(0) \right|.$$

Intuitively this can be seen as disregarding the information about whether the manifold is curved in a convex or concave way along certain tangent directions. In later chapters we will see that we do not need this information to construct good asymptotic upper bounds on meshing errors. We call this the *convexification* step.

Let

$$C(\eta_p, \nu_p) = \sum_{i=1}^{c} C(\eta_p, \nu_p^i)$$

be the *curvature matrix* corresponding to the adapted frame (η_p, ν_p). This matrix is positive semidefinite, since it is the sum of positive semidefinite matrices.

We will need a positive definite matrix in the sequel, so instead of using $C(\eta_p, \nu_p)$ directly we will use a matrix

$$C(\eta_p, \nu_p) + \delta I$$

for some small $\delta > 0$.

A curvature matrix $C(\eta_p, \nu_p)$ depends on the choice of adapted frame (η_p, ν_p). In the following two lemmas we will show that the dependence on the tangent frame η_p is just a consequence of working in local coordinates. By this we mean that curvature matrices which have the same normal frame are similar, where two matrices A and B are *similar* if there exists a matrix S such that $A = S^{-1}BS$. However, the dependence on the normal frame is important: By choosing different normal frames in general we obtain curvature matrices with for example different eigenvalues.

Lemma 4.6. *If $g : \mathbb{R}^n \to \mathbb{R}$ is a smooth map and $h(x) = g(Ax)$ where $A \in \mathbb{R}^{n\times n}$, then*

$$Hh(0) = A^T Hg(0)A.$$

Proof. The Taylor expansion of g around 0 is given by

$$g(x) = g(0) + Dg(0)x + \frac{1}{2}x^T Hg(0)x + o(||x||^2),$$

so

$$h(x) = g(Ax) = g(0) + Dg(0)Ax + \frac{1}{2}x^T A^T Hg(0)Ax + o(||x||^2),$$

hence

$$Hh(0) = A^T Hg(0)A.$$

□

Lemma 4.7. *If η_p and ζ_p are orthonormal tangent frames, then there is an $R \in O(\mathbb{R}^d)$ such that $\eta_p = R\zeta_p$, and*

$$C(\eta_p, \nu_p) = RC(\zeta_p, \nu_p)R^T.$$

Proof. By Lemma 4.6,

$$Hf(\eta_p, \nu_p^i)(0) = Hf(R\zeta_p, \nu_p^i)(0) = H(f(\zeta_p, \nu_p^i) \circ R^T)(0) = RHf(\zeta_p, \nu_p^i)(0)R^T.$$

Let $Hf(\zeta_p, \nu_p^i)(0) = Q_i^T D_i Q_i$ be a diagonalization, then

$$Hf(\eta_p, \nu_p^i)(0) = RQ_i^T D_i Q_i R^T$$

is a diagonalization, so

$$C(\eta_p, \nu_p^i) = RQ_i^T |D_i| Q_i R^T = RC(\zeta_p, \nu_p^i)R^T.$$

□

Curvature numbers. Let

$$c_\nu^\delta : M \to \mathbb{R}, \quad p \mapsto \det(C(\eta_p, \nu_p) + \delta I),$$

for some orthonormal tangent frame η_p of M at p. Lemma 4.7 shows that the right hand side is independent of the choice of tangent frame η_p. We call $c_\nu^\delta(p)$ a *curvature number* of M at p. Denote the curvature number $c_\nu^\delta(p)$ by $c_\nu(p)$.

Let ν_p be an orthonormal normal frame of M at p and denote by $O(c)$ the group of orthogonal transformations on N_pM. For $\omega \in O(c)$, let $\omega(\nu_p) = \{\omega(\nu_p^1), \ldots, \omega(\nu_p^c)\}$. Since the map $O(c) \to \mathbb{R}$, $\omega \mapsto c_{\omega(\nu)}(p)$ is continuous and since $O(c)$ is compact there exists an orthonormal frame ν_p for each p such that $c_\nu(p)$ is minimal over all orthonormal normal frames at p, that is,

$$c_\nu(p) = \min\{c_\tau(p) \mid \tau \text{ is an orthonormal normal frame at } p\}.$$

We denote the curvature number corresponding to a curvature minimizing frame by $c(p)$.

4.4 Curvature Tensor Fields

In this section we will use the positive definite curvature matrices to construct Riemannian tensor fields which we will call curvature tensor fields. Let M be a d-manifold embedded in $\mathbb{R}^{d+c}$. Given a point $p \in M$, an orthonormal normal frame ν_p at p, and some number $\delta > 0$, let

$$(q_\nu^\delta)_p : T_pM \times T_pM \to \mathbb{R}$$

be the map defined by

$$(q_\nu^\delta)_p(u, v) = [u]_{\eta_p}^T (C(\eta_p, \nu_p) + \delta I)[v]_{\eta_p},$$

where η_p is some orthonormal tangent frame at p. We start by showing that the right hand side of this equation is independent of the choice of orthonormal tangent frame η_p.

Lemma 4.8. *If M is a smooth d-manifold embedded in $\mathbb{R}^{d+c}$, ν_p is an orthonormal normal frame at p, and η_p and ζ_p are orthonormal tangent frames at p, then*

$$[u]_{\eta_p}^T (C(\eta_p, \nu_p) + \delta I)[v]_{\eta_p} = [u]_{\zeta_p}^T (C(\zeta_p, \nu_p) + \delta I)[v]_{\zeta_p}.$$

Proof. Let $R \in O(\mathbb{R}^d)$ be the orthogonal transformation such that $\eta_p = R\zeta_p$. By Lemma 4.7 we have that

$$C(\eta_p, \nu_p) = RC(\zeta_p, \nu_p)R^T,$$

hence

$$\begin{aligned}
[u]_{\eta_p}^T (C(\eta_p, \nu_p) + \delta I)[v]_{\eta_p} &= [u]_{\eta_p}^T C(\eta_p, \nu_p)[v]_{\eta_p} + \delta [u]_{\eta_p}^T [v]_{\eta_p} \\
&= [u]_{\zeta_p}^T R^T R C(\zeta_p, \nu_p) R^T R [v]_{\eta_p} + \delta [u]_{\zeta_p}^T R^T R [v]_{\zeta_p} \\
&= [u]_{\zeta_p}^T C(\zeta_p, \nu_p)[v]_{\zeta_p} + \delta [u]_{\zeta_p}^T [v]_{\zeta_p} \\
&= [u]_{\zeta_p}^T (C(\zeta_p, \nu_p) + \delta I)[v]_{\zeta_p}.
\end{aligned}$$

□

It follows from the linearity of the coordinate maps $v \mapsto [v]_{\eta_p}$ that the maps $(q^\delta_\nu)_p$ are bilinear. Hence we have that for each $p \in M$, the map $(q^\delta_\nu)_p$ is a covariant 2-tensor. We will now show that q^δ_ν is a covariant 2-tensor field of class C^0 by showing that for any two C^0 vector fields u and v on M, the map

$$M \to \mathbb{R},\ p \mapsto (q^\delta_\nu)_p(u_p, v_p)$$

is of class C^0.

Lemma 4.9. *If M is a smooth d-manifold embedded in $\mathbb{R}^{d+c}$, ν is a nonoriented orthonormal normal frame field, and $\delta > 0$, then q^δ_ν is a covariant 2-tensor field of class C^0.*

Proof. Let $p \in M$ and let u and v be vector field of class C^0 on M. Let η be an orthonormal tangent frame field in an open neighbourhood U_p of p. Then $p \mapsto [u_p]_{\eta_p}$ and $p \mapsto [v_p]_{\eta_p}$ are continuous maps from U_p to $\mathbb{R}^d$. The entries of $C(\eta_p, \nu_p)$ depend continuously on p, so the product $[u]^T_{\eta_p} C(\eta_p, \nu_p)[v]_{\eta_p}$ is continuous in p. □

Since we have constructed the matrices $C(\eta_p, \nu_p) + \delta I$ to be positive definite, we have the immediate corollary that q^δ_ν is a Riemannian tensor field.

Corollary 4.10. *If M is a smooth d-manifold embedded in $\mathbb{R}^{d+c}$, ν is a nonoriented orthonormal normal frame field, and $\delta > 0$, then q^δ_ν is a Riemannian tensor field of class C^0.*

Smoothing. Although a Riemannian tensor field of class C^0 is well defined, it is customary for a Riemannian tensor field to be of class C^∞. And although it is sufficient for q^δ_ν to be of class C^0 in most places in this report, there is a single place where it needs to be of class C^2, hence we will discuss the smoothing of tensor fields.

Lemma 4.11. *If M is a compact d-manifold of class C^k endowed with a Riemannian tensor field q of class C^0, then for each $\delta > 0$ there exists a Riemannian tensor field g of class C^k on M and an atlas $\{x_\alpha : V_\alpha \to \mathbb{R}^d \mid \alpha \in A\}$ of M such that*

$$\left| q_p\left(\frac{\partial}{\partial x^i_\alpha}\Big|_p, \frac{\partial}{\partial x^j_\alpha}\Big|_p\right) - g_p\left(\frac{\partial}{\partial x^i_\alpha}\Big|_p, \frac{\partial}{\partial x^j_\alpha}\Big|_p\right)\right| < \delta$$

for all $\alpha \in A$, $p \in V_\alpha$, and $1 \le i, j \le d$.

Proof. Since M is compact, there exists a finite subset $B \subset A$ such that

$$\{x_\alpha : V_\alpha \to \mathbb{R}^d \mid \alpha \in B\}$$

is an atlas of B. For each $\beta \in B$, let W_β be a compact subset of V_β and X_β an open subset of W_β such that $\{X_\beta \mid \beta \in B\}$ is an open cover of M.

Let $\beta \in B$ and let $\delta > 0$. For all $1 \le i, j \le d$, the functions

$$W_\beta \to \mathbb{R}, p \mapsto g_p\left(\frac{\partial}{\partial x^i_\beta}\Big|_p, \frac{\partial}{\partial x^j_\beta}\Big|_p\right)$$

are continuous and have compact domain. So, for each $\lambda > 0$, by the Weierstrass approximation theorem, there exist smooth functions

$$\phi^{i,j}_\beta : W_\beta \to \mathbb{R}$$

such that

$$\left| g_p \left(\frac{\partial}{\partial x^i_\beta}\Big|_p, \frac{\partial}{\partial x^j_\beta}\Big|_p \right) - \phi^{i,j}_\beta(p) \right| < \lambda\delta,$$

for all $1 \le i \le j \le d$. Let $\phi^{j,i}_\beta = \phi^{i,j}_\beta$ for all $1 \le i < j \le d$. We define g_β by

$$(g_\beta)_p \left(\frac{\partial}{\partial x^i_\beta}\Big|_p, \frac{\partial}{\partial x^j_\beta}\Big|_p \right) = \phi^{i,j}_\beta(p),$$

for all $1 \le i, j \le d$ and all $p \in W_\beta$. The form g_β is symmetric and if λ is sufficiently small then g_β is positive definite. Since the functions $\phi^{i,j}_\beta$ are smooth and since the vector fields

$$p \mapsto \frac{\partial}{\partial x^i_\beta}\Big|_p$$

are of class C^k, g_β is a Riemannian tensor field of class C^k on $X_\beta \subset W_\beta$.

Since $\{X_\beta \mid \beta \in B\}$ is a finite open cover of M, we can use a partition of unity to obtain a Riemannian tensor field g on M from the tensor fields g_β. And if $\lambda > 0$ is sufficiently small then this tensor field g satisfies

$$\left| q_p \left(\frac{\partial}{\partial x^i_\alpha}\Big|_p, \frac{\partial}{\partial x^j_\alpha}\Big|_p \right) - g_p \left(\frac{\partial}{\partial x^i_\alpha}\Big|_p, \frac{\partial}{\partial x^j_\alpha}\Big|_p \right) \right| < \delta$$

for all $\alpha \in A$, $p \in V_\alpha$, and $1 \le i, j \le d$. □

Let ν be a nonoriented orthonormal normal frame field of M and let $\delta > 0$. We can use the previous lemma to obtain a smooth Riemannian tensor field g^δ_ν of M such that there exists an atlas $\{x_\alpha : V_\alpha \to \mathbb{R}^d \mid \alpha \in A\}$ of M, such that

$$\left| (q^\delta_\nu)_p \left(\frac{\partial}{\partial x^i_\alpha}\Big|_p, \frac{\partial}{\partial x^j_\alpha}\Big|_p \right) - (g^\delta_\nu)_{p_\alpha} \left(\frac{\partial}{\partial x^i_\alpha}\Big|_p, \frac{\partial}{\partial x^j_\alpha}\Big|_p \right) \right| < \delta$$

for all $\alpha \in A$, all $p_\alpha \in V_\alpha$, and all $1 \le i, j \le d$. When δ gets small, q^δ_ν will get closer to g^δ_ν and at the same time the matrix $C(\eta_p, \nu_p) + \delta I$, which is used in the definition of g^δ_ν, gets closer to $C(\eta_p, \nu_p)$. This means in particular that $\det((q^\delta_\nu)_p) \to c^\delta_\nu(p)$ as $\delta \to 0$.

4.5 Curvature Metrics and Measures

As explained in Section 2.4, the Riemannian tensor field q^δ_ν induces the metric

$$\begin{aligned} \mathrm{d}^\delta_\nu(p, q) &= \inf_{\gamma \in \Gamma(p,q)} \int_0^1 \sqrt{(q^\delta_\nu)_{\gamma(t)}(\gamma'(t), \gamma'(t))}\, \mathrm{d}t \\ &= \inf_{\gamma \in \Gamma(p,q)} \int_0^1 \left\| \sqrt{C(\eta_{\gamma(t)}, \nu_{\gamma(t)}) + \delta I}\, [\gamma'(t)]_{\eta_{\gamma(t)}} \right\| \mathrm{d}t, \end{aligned}$$

for $p, q \in M$, where for each p, η_p is some orthonormal tangent frame at $p \in M$. We call such a metric a *curvature metric*. For small $\delta > 0$, this metric satisfies the property that distances are large in regions of high curvature and vice versa.

We define a *curvature measure*, denoted by μ_ν^δ, to be the measure induced by the Riemannian structure q_ν^δ. From the definition of a curvature number it follows that

$$\mu_\nu^\delta(V) = \int_V c_\nu^\delta(p)\,\mathrm{d}\sigma(p),$$

for Borel sets $V \subset M$.

If $\delta = 0$, then in general q_ν^δ is not a Riemannian tensor field, since it is not necessarily positive definite everywhere. Therefore, d_ν^δ is not defined for $\delta = 0$. The curvature numbers $c_\nu^\delta(p)$, however, are defined for $\delta = 0$. Hence we can use the curvature numbers to define

$$\mu_\nu(V) = \mu_\nu^0(V) = \int_V c_\nu^0(p)\,\mathrm{d}\sigma(p)$$

for Borel sets $V \subset M$. Note that $\mu_\nu^{\delta_1}(V) \leq \mu_\nu^{\delta_2}(V)$ if $\delta_1 \leq \delta_2$, and

$$\lim_{\delta\downarrow 0} \mu_\nu^\delta(V) = \mu_\nu^0(V).$$

Given an embedded manifold with an nonoriented orthonormal normal frame field and a value $\delta > 0$, we obtain in this way a metric measure space $(M, \mathrm{d}_\nu^\delta, \mu_\nu^\delta)$ and a measure space (M, μ_ν), such that in particular $\mu_\nu^\delta(M) \downarrow \mu_\nu^0(M)$ as $\delta \downarrow 0$.

4.6 Specialization to Hypersurfaces

A *d-hypersurface* is a d-manifold embedded in $\mathbb{R}^{d+1}$. Since a differentiable hypersurface M is orientable, M admits precisely two global unit normal fields [14]. Given such a unit normal field ν^1, we have an orthonormal normal frame field $\nu = \{\nu^1\}$. The curvature matrix at a point $p \in M$, with respect to the normal frame field ν and some orthonormal tangent frame η_p, is given by

$$C(\eta_p, \nu_p) = C(\eta_p, \nu_p^1) = \left| Hf(\eta_p, \nu_p^i)(0) \right|.$$

The other unit normal field on M is given by $-\nu^1$, and we have that $f(\eta_p, \nu_p^i) = -f(\eta_p, -\nu_p^i)$. Therefore, in the case of a hypersurface, the curvature matrix $C(\eta_p, \nu_p)$ is independent of the normal frame field. We will drop the normal frame in the notation to obtain curvature matrices $C(\eta_p)$, curvature numbers c^δ, curvature tensor fields q^δ, curvature metrics d^δ, and curvature measures μ^δ. As noted in the introduction of this chapter, the construction of curvature tensor fields is a generalizations of a method in [3] to higher codimension. As expected, the specialization of these curvature tensor fields to codimension one yields the same forms, metrics, and measures as in the referenced paper.

In Lemma 4.7 we established that the eigenvalues of a curvature matrix do not depend on the choice of orthonormal tangent frame η_p. Consequently, in the codimension one case, the eigenvalues of a curvature matrix at p are canonical quantities. In fact, these eigenvalues are the absolute values of the principal curvatures $\kappa_1(p), \ldots, \kappa_d(p)$ of M. It follows that their product, the curvature number $c^0 = \det(C_{\eta_p})$, is equal to the absolute value of the Gaussian curvature $K(p)$. This means that in the case of hypersurfaces the curvature measure simplifies to

$$\mu^0(M) = \int_M \sqrt{|K(p)|}\,\mathrm{d}\sigma(p),$$

and is also known as the *root curvature measure*.

We have seen that the eigenvectors of a curvature matrix in codimension one are the principal curvatures. Similarly, we have that if $v_1, \ldots, v_d$ are the eigenvectors of a curvature matrix $C(\eta_p)$, then the vectors $w_1, \ldots, w_d$ for which $v_1 = [w_1]_{\eta_p}, \ldots, v_d = [w_d]_{\eta_p}$ are the principal directions of M at p. Note that, by Lemma 4.7, these vectors $w_1, \ldots, w_d$ are independent of the choice of orthonormal tangent frame η_p.

4.7 Bounds on the Curvature Metrics

Given a nonoriented orthonormal normal frame field ν on M, and some $\delta > 0$, we can approximate a metric $\mathrm{d}^\delta_\nu(p, q)$ in the vicinity of a point $r \in M$ by using projections on the tangent plane at r and by using a curvature matrix at r. The following two lemmas reformulate the first part of Lemma 4.1 of Clarkson [3] in our notation. We also provide detailed proofs.

Lemma 4.12. *If M is a compact smooth d-manifold embedded in $\mathbb{R}^{d+c}$, $r \in M$, ν a nonoriented orthonormal normal frame field of M, and η_r an orthonormal tangent frame at r, then there is an open neighbourhood V of r, such that*

$$\mathrm{d}^\delta_\nu(p, q) \leq \lambda \mathrm{d}_E(\sqrt{C(\eta_r, \nu_r)}\tau_{\eta_r} p, \sqrt{C(\eta_r, \nu_r)}\tau_{\eta_r} q),$$

for all $p, q \in V$, and all $\delta > 0$ sufficiently small.

Proof. Assume, without loss of generality, that r is the origin and $T_r M$ is the

$$x_{d+1} = \ldots = x_{d+c} = 0$$

plane.

If V is sufficiently small, then the points of $\psi(t) = (1 - t)\tau_{\eta_r} p + t\tau_{\eta_r} q$ are contained in the image of $\tau_{\eta_r} U_r$ for all $t \in [0, 1]$. Hence,

$$\sigma(t) = \big(\psi(t), f(\eta_r, \nu_r^1)(\psi(t)), \ldots, f(\eta_r, \nu_r^c)(\psi(t))\big)$$

is an element of $\Gamma(p, q)$. Note that

$$\big[\sigma'(t)\big]_{\eta_r} = \psi'(t) = \tau_{\eta_r} q - \tau_{\eta_r} p.$$

Let V be a small open neighbourhood of r. The frame η_r can be extended to an orthonormal tangent frame field on V. Furthermore, $\sigma(t)$ stays close to r for all $t \in [0, 1]$. This means that the adapted frame $(\eta_{\sigma(t)}, \nu_{\sigma(t)})$ stays close to the frame (η_r, ν_r) and the matrix $C(\eta_{\sigma(t)}, \nu_{\sigma(t)})$ stays close to $C(\eta_r, \nu_r)$ for all $t \in [0, 1]$. Therefore,

$$\left\| \sqrt{C(\eta_{\sigma(t)}, \nu_{\sigma(t)})}\, \big[\sigma'(t)\big]_{\eta_{\sigma(t)}} \right\| \leq \sqrt{\lambda} \left\| \sqrt{C(\eta_r, \nu_r)}\, \big[\sigma'(t)\big]_{\eta_r} \right\|,$$

for all $t \in [0, 1]$, by taking V small enough.

Hence, for $\delta > 0$ and V both sufficiently small, we have

$$\begin{aligned}
\mathrm{d}_\nu^\delta(p,q) &= \inf_{\gamma \in \Gamma(p,q)} \int_0^1 \left\| \sqrt{C(\eta_{\gamma(t)}, \nu_{\gamma(t)}) + \delta I} \left[\gamma'(t)\right]_{\eta_{\gamma(t)}} \right\| \mathrm{d}t \\
&\leq \sqrt{\lambda} \inf_{\gamma \in \Gamma(p,q)} \int_0^1 \left\| \sqrt{C(\eta_{\gamma(t)}, \nu_{\gamma(t)})} \left[\gamma'(t)\right]_{\eta_{\gamma(t)}} \right\| \mathrm{d}t \\
&\leq \sqrt{\lambda} \int_0^1 \left\| \sqrt{C(\eta_{\sigma(t)}, \nu_{\sigma(t)})} \left[\sigma'(t)\right]_{\eta_{\sigma(t)}} \right\| \mathrm{d}t \\
&\leq \sqrt{\lambda} \int_0^1 \sqrt{\lambda} \left\| \sqrt{C(\eta_r, \nu_r)} \left[\sigma'(t)\right]_{\eta_r} \right\| \mathrm{d}t \\
&= \lambda \int_0^1 \left\| \sqrt{C(\eta_r, \nu_r)} \psi'(t) \right\| \mathrm{d}t \\
&= \lambda \int_0^1 \left\| \sqrt{C(\eta_r, \nu_r)} (\tau_{\eta_r} q - \tau_{\eta_r} p) \right\| \mathrm{d}t \\
&= \lambda \left\| \sqrt{C(\eta_r, \nu_r)} (\tau_{\eta_r} q - \tau_{\eta_r} p) \right\| \\
&= \lambda \mathrm{d}_E(\sqrt{C(\eta_r, \nu_r)} \tau_{\eta_r} p, \sqrt{C(\eta_r, \nu_r)} \tau_{\eta_r} q).
\end{aligned}$$

□

Since a metric d_ν^δ is a length metric, one would expect that we need a variational argument to bound the d_ν^δ distance from below. Instead of using a variational argument, we reduce this problem to one in a scaled Euclidean space, for which we already know that the length minimizing curves are straight line segments.

Lemma 4.13. *If M is a compact smooth d-manifold embedded in $\mathbb{R}^{d+c}$, $\lambda > 0$, $r \in M$, ν a nonoriented orthonormal normal frame field of M, and η_r an orthonormal tangent frame at r, then there is an open neighbourhood V of r, such that*

$$\mathrm{d}_E(\sqrt{C(\eta_r, \nu_r)} \tau_{\eta_r} p, \sqrt{C(\eta_r, \nu_r)} \tau_{\eta_r} q) \leq \lambda \mathrm{d}_\nu^\delta(p,q),$$

for all $p, q \in V$, and all $\delta > 0$.

Proof. Assume, without loss of generality, that r is the origin and $T_r M$ is the

$$x_{d+1} = \ldots = x_{d+c} = 0$$

plane.

For all $\delta > 0$, we have

$$\begin{aligned}
\mathrm{d}_\nu^\delta(p,q) &= \inf_{\gamma \in \Gamma(p,q)} \int_0^1 \left\| \sqrt{C(\eta_{\gamma(t)}, \nu_{\gamma(t)}) + \delta I} \left[\gamma'(t)\right]_{\eta_{\gamma(t)}} \right\| \mathrm{d}t \\
&\geq \inf_{\gamma \in \Gamma(p,q)} \int_0^1 \left\| \sqrt{C(\eta_{\gamma(t)}, \nu_{\gamma(t)})} \left[\gamma'(t)\right]_{\eta_{\gamma(t)}} \right\| \mathrm{d}t
\end{aligned}$$

Since M is compact, there is a $\sigma \in \Gamma(p,q)$ such that

$$\inf_{\gamma \in \Gamma(p,q)} \int_0^1 \left\| \sqrt{C(\eta_{\gamma(t)}, \nu_{\gamma(t)})} \left[\gamma'(t)\right]_{\eta_{\gamma(t)}} \right\| \mathrm{d}t = \int_0^1 \left\| \sqrt{C(\eta_{\sigma(t)}, \nu_{\sigma(t)})} \left[\sigma'(t)\right]_{\eta_{\sigma(t)}} \right\| \mathrm{d}t.$$

Let V be a small open neighbourhood of r. The frame η_r can be extended to an orthonormal tangent frame field on V. Furthermore, $\sigma(t)$ stays close to r for all $t \in [0,1]$. This means that the adapted frame $(\eta_{\sigma(t)}, \nu_{\sigma(t)})$ stays close to the frame (η_r, ν_r) and the matrix $C(\eta_{\sigma(t)}, \nu_{\sigma(t)})$ stays close to $C(\eta_r, \nu_r)$ for all $t \in [0,1]$. Therefore,

$$\left\| \sqrt{C(\eta_{\sigma(t)}, \nu_{\sigma(t)})} \left[\sigma'(t)\right]_{\eta_{\sigma(t)}} \right\| \geq \frac{1}{\lambda} \left\| \sqrt{C(\eta_r, \nu_r)} \left[\sigma'(t)\right]_{\eta_r} \right\|,$$

for all $t \in [0,1]$, by choosing V small with respect to λ.

We also have that, by the implicit function theorem,

$$\sigma(t) = \left(\psi(t), f(\eta_r, \nu_r^1)(\psi(t)), \ldots, f(\eta_r, \nu_r^c)(\psi(t))\right)$$

for some smooth curve $\psi(t) : [0,1] \to \mathbb{R}^d$. Note that $\left[\sigma'(t)\right]_{\eta_r} = \psi'(t)$.

Hence we have,

$$\begin{aligned}
\int_0^1 \left\| \sqrt{C(\eta_{\sigma(t)}, \nu_{\sigma(t)})} \left[\sigma'(t)\right]_{\eta_{\sigma(t)}} \right\| \mathrm{d}t &\geq \frac{1}{\lambda} \int_0^1 \left\| \sqrt{C(\eta_r, \nu_r)} \left[\sigma'(t)\right]_{\eta_r} \right\| \mathrm{d}t \\
&= \frac{1}{\lambda} \int_0^1 \left\| \sqrt{C(\eta_r, \nu_r)} \psi'(t) \right\| \mathrm{d}t \\
&\geq \frac{1}{\lambda} \inf_{\theta \in \Gamma(\tau_{\eta_r} p, \tau_{\eta_r} q)} \int_0^1 \left\| \sqrt{C(\eta_r, \nu_r)} \theta'(t) \right\| \mathrm{d}t
\end{aligned}$$

This last integral is the length of the curve θ in a Euclidean vector space which is scaled along the coordinate axes. Hence the optimal curve exists, that is, the infimum is attained, and is given by $\phi(t) = (t-1)\tau_{\eta_r} p + t\tau_{\eta_r} q$. This gives us

$$\begin{aligned}
\inf_{\theta \in \Gamma(\tau_{\eta_r} p, \tau_{\eta_r} q)} \int_0^1 \left\| \sqrt{C(\eta_r, \nu_r)} \theta'(t) \right\| \mathrm{d}t &= \int_0^1 \left\| \sqrt{C(\eta_r, \nu_r)} \phi'(t) \right\| \mathrm{d}t \\
&= \int_0^1 \left\| \sqrt{C(\eta_r, \nu_r)} (\tau_{\eta_r} q - \tau_{\eta_r} p) \right\| \mathrm{d}t \\
&= \left\| \sqrt{C(\eta_r, \nu_r)} (\tau_{\eta_r} q - \tau_{\eta_r} p) \right\| \\
&= \mathrm{d}_E(\sqrt{C(\eta_r, \nu_r)} \tau_{\eta_r} p, \sqrt{C(\eta_r, \nu_r)} \tau_{\eta_r} q).
\end{aligned}$$

So, we have

$$\mathrm{d}_E(\sqrt{C(\eta_r, \nu_r)} \tau_{\eta_r} p, \sqrt{C(\eta_r, \nu_r)} \tau_{\eta_r} q) \leq \lambda \mathrm{d}_\nu^\delta(p, q).$$

□

4.8 Deviation from Linearity

If $f(x)$ is a function with $f'(a) = 0$, then the deviation from linearity in the vicinity of a is approximately $\frac{1}{2} f''(a)(x-a)^2$. The following lemma is related to this idea in the sense that it expresses the deviation from linearity in a quadratic formula involving the second derivatives and the deviation along the tangent plane. Because we use curvature matrices, which are constructed using the convexified Hessian matrices, we only provide an upper bound.

This lemma is a generalization to higher codimension of a part of Lemma 4.1 of [3]. We give a detailed proof and show that an additional factor $\frac{1}{2}$ can be added to the right hand side.

Lemma 4.14. *Let M be a compact smooth d-manifold embedded in $\mathbb{R}^{d+c}$. For each $\lambda > 1$, $r \in M$, and adaptive coordinate system (η_r, ν_r) of M at r, there is an open neighbourhood V of r, such that*

$$\mathrm{d}_E(p, T_q M) \leq \frac{\lambda}{2} \mathrm{d}_E(\sqrt{C(\eta_r, \nu_r)} \tau_{\eta_r} p, \sqrt{C(\eta_r, \nu_r)} \tau_{\eta_r} q)^2, \tag{4.1}$$

for all $p, q \in V$.

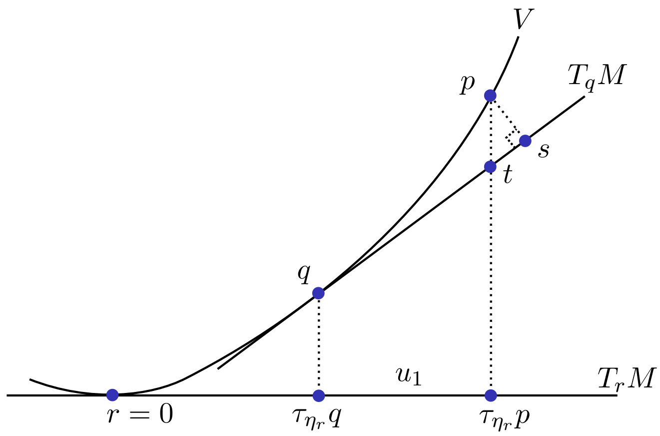

Figure 4.2: Figure for the proof of Lemma 4.14 for the $d = c = 1$ case.

Proof. Assume, without loss of generality, that r is the origin and $\eta_r^i = e_i$ and $\nu_r^j = e_{d+j}$ for all $1 \leq i \leq d$ and $1 \leq j \leq c$. Let $\pi : \mathbb{R}^{d+c} \to \mathbb{R}^d$ be the map that drops the last c coordinates. Given the above assumptions, we have, $\pi|_{U_r} = \tau_{\eta_r}$.

Let P_i be the

$$x_{d+1} = \ldots = x_{d+i-1} = x_{d+i+1} = \ldots = x_{d+c} = 0$$

plane, and let $\perp_i$ the projection map on the plane P_i, for $i = 1, \ldots, c$. In the $c = 1$ case these definitions entail that P_1 is the full space $\mathbb{R}^{d+c}$ and $\perp_1 = \mathrm{Id}$. We have

$$\tau_{\eta_r} p = \pi p = \pi \perp_1 p = \ldots = \pi \perp_c p \quad \text{and} \quad \tau_{\eta_r} q = \pi q = \pi \perp_1 q = \ldots = \pi \perp_c q.$$

Let s be the projection of p on the affine subspace $T_q M$. Let t be the point on the affine subspace $T_q M$ such that $\pi p = \pi t$; such a unique point t exists when V is sufficiently small, by smoothness of the manifold.

Then we have

$$\mathrm{d}_E(p, T_q M) = \mathrm{d}_E(p, s) \leq \mathrm{d}_E(p, t),$$

where the inequality follows from triangle inequality.

We use a linear approximation to obtain

$$\begin{aligned}
\mathrm{d}_E(\perp_i p, \perp_i t) &= \left| f(\eta_r, \nu_r^i)(\tau_{e_r} p) - \left(f(\eta_r, \nu_r^i)(\tau_{\eta_r} q) + Df(\eta_r, \nu_r^i)(\tau_{\eta_r} q)(\tau_{\eta_r} p - \tau_{\eta_r} q) \right) \right| \\
&= \frac{1}{2} \left| (\tau_{\eta_r} p - \tau_{\eta_r} q)^T Hf(\eta_r, \nu_r^i)(u_i)(\tau_{\eta_r} p - \tau_{\eta_r} q) \right| \\
&\leq \frac{\lambda}{2} \left| (\tau_{\eta_r} p - \tau_{\eta_r} q)^T Hf(\eta_r, \nu_r^i)(0)(\tau_{\eta_r} p - \tau_{\eta_r} q) \right|,
\end{aligned}$$

where the second equality holds, by the Lagrange remainder theorem, for some u_i on the line segment between $\tau_{\eta_r} q$ and $\tau_{\eta_r} p$; the inequality holds for V sufficiently small, by smoothness of the Hessian.

Since $C(\eta_r, \nu_r^i)$ and $C(\eta_r, \nu_r^i) - Hf(\eta_r, \nu_r^i)(0)$ are positive semidefinite, we have

$$\begin{aligned}
\left|(\tau_{\eta_r} p - \tau_{\eta_r} q)^T Hf(\eta_r, \nu_r^i)(0)(\tau_{\eta_r} p - \tau_{\eta_r} q)\right| &\leq \left|(\tau_{\eta_r} p - \tau_{\eta_r} q)^T C(\eta_r, \nu_r^i)(\tau_{\eta_r} p - \tau_{\eta_r} q)\right| \\
&= (\tau_{\eta_r} p - \tau_{\eta_r} q)^T C(\eta_r, \nu_r^i)(\tau_{\eta_r} p - \tau_{\eta_r} q).
\end{aligned}$$

This yields

$$\begin{aligned}
\mathrm{d}_E(p, T_q M) &\leq \sum_{i=1}^{c} \mathrm{d}_E(\perp_i p, \perp_i t) \\
&\leq \sum_{i=1}^{c} \frac{1}{2} C(\tau_{\eta_r} p - \tau_{\eta_r} q)^T C(\eta_r, \nu_r^i)(\tau_{\eta_r} p - \tau_{\eta_r} q) \\
&= \frac{1}{2} C(\tau_{\eta_r} p - \tau_{\eta_r} q)^T \left(\sum_{i=1}^{c} C(\eta_r, \nu_r^i)\right)(\tau_{\eta_r} p - \tau_{\eta_r} q) \\
&= \frac{1}{2} C(\tau_{\eta_r} p - \tau_{\eta_r} q)^T C(\eta_r, \nu_r)(\tau_{\eta_r} p - \tau_{\eta_r} q) \\
&= \frac{1}{2} C \left\| \sqrt{C(\eta_r, \nu_r)}(\tau_{\eta_r} p - \tau_{\eta_r} q)\right\|_2^2 \\
&= \frac{1}{2} C \mathrm{d}_E(\sqrt{C(\eta_r, \nu_r)}\tau_{\eta_r} p, \sqrt{C(\eta_r, \nu_r)}\tau_{\eta_r} q)^2,
\end{aligned}$$

where we use the triangle inequality in the first inequality. □

The following corollary follows by combining the results of Lemma 4.13 and Lemma 4.14 and will be important since it allows us to use curvature tensor fields to give quadratic upper bounds on the deviation from linearity.

Corollary 4.15. *Let M be a compact smooth d-manifold embedded in $\mathbb{R}^{d+c}$, ν a nonoriented orthonormal normal frame field of M, and $\delta > 0$. If $r \in M$ and $\lambda > 1$, then there is an open neighbourhood V of r such that*

$$\mathrm{d}_E(p, T_q M) \leq \frac{\lambda}{2} \mathrm{d}_u^\delta(p, q)^2,$$

for all $p, q \in V$.

Chapter 5

Meshing of Embedded Manifolds

In this chapter we discuss the problem of finding meshes for compact differentiable manifolds embedded in Euclidean spaces. Given a Riemannian structure on the manifold, we will construct meshes that satisfy certain properties with respect to the metric induced by this structure. In Section 6.1 we will use this to find upper bounds on the Hausdorff distance between the manifolds and the meshes by using curvature tensor fields, as defined in Section 4.4, as the Riemannian structures. However, in this chapter we only assume that the manifold is equipped with an arbitrary Riemannian structure, so we will not need any of the material from Chapter 4.

We will use Delaunay triangulations to obtain the desired meshes. These manifold triangulations are discussed in Section 5.1 and Delaunay triangulations are discussed in Section 5.2. In Section 5.4 we use results from Leibon and Letscher [11] to show that for each sufficiently dense net there exists a Delaunay triangulation with has as vertex set this net. Finally, in Section 5.5 we discuss the construction of a mesh from a triangulation, and we show that there can be local and global obstructions for this to be possible. We show that it is possible to obtain a mesh from a Delaunay triangulation whose vertex set is an ϵ-net with ϵ sufficiently small.

5.1 Triangulations

In this section we discuss topological triangulations, to which we will refer as triangulations. To be able to give a definition of a triangulation we need the concept of simplicial complexes, which we will introduce first.

Let $v_0, \ldots, v_d$ be a set of *affinely independent* points in $\mathbb{R}^n$, that is, let $v_0, \ldots, v_d$ be points in $\mathbb{R}^d$ such that the points $v_1 - v_0, \ldots, v_d - v_0$ are linearly independent. The convex hull

$$[v_0, \ldots, v_d]$$

of such a set of points is called a *d-simplex*. The points $v_0, \ldots, v_d$ are called the *vertices* of this simplex. The convex hull of a subset of k points from $\{v_0, \ldots, v_d\}$ is called a k-face of $[v_0, \ldots, v_d]$.

A *simplicial complex* in $\mathbb{R}^n$ is a collection of simplices in $\mathbb{R}^n$ which contains each face of each simplex contained in it, and where the intersection of two simplices is either empty or is a face of both. In Figure 5.1 we depict a collection of simplicies which is not a simplicial complex since not all faces of the triangle are contained in the collection. The collection of

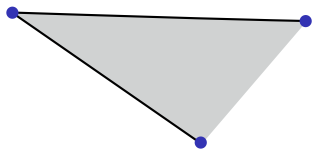

Figure 5.1: Missing face.

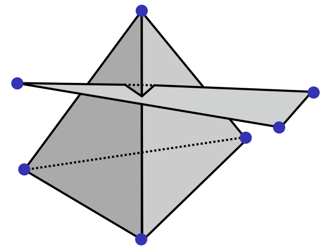

Figure 5.2: Nonempty intersection which is not a face.

simplices in Figure 5.2 is not a simplicial complex since the nonempty intersection of some simplices is not a shared face. The union of the simplices in a simplicial complex $\mathcal{A}$ is denoted by $|\mathcal{A}|$, and we call this the *carrier* of the simplicial complex. Two simplicial complexes $\mathcal{A}$ and $\mathcal{B}$ are *isomorphic* if there exists a bijection $\phi : \mathcal{A} \to \mathcal{B}$ such that $F \in \mathcal{A}$ is a face of $A \in \mathcal{A}$ if and only if $\phi(F) \in \mathcal{B}$ is a face of $\phi(A) \in \mathcal{B}$. Isomorphic simplicial complexes are said to have the same *combinatorial structure*. The carriers of isomorphic simplicial complexes are homeomorphic.

A *triangulation* of a topological space X is a tuple $(\mathcal{X}, h)$ which consists of a simplicial complex $\mathcal{X}$ and a homeomorphism $h : |\mathcal{X}| \to X$. We call $h(V(\mathcal{X}))$ the *vertex set* of the triangulation. Two triangulations $(\mathcal{X}_1, h_1)$ and $(\mathcal{X}_2, h_2)$ of a topological space X are *isomorphic* if they have equal vertex sets and if $\mathcal{X}_1$ and $\mathcal{X}_2$ have the same combinatorial structure. In [17, Chapter IV, Part B] it is shown that every smooth manifold admits a triangulation.

5.2 Delaunay Triangulations

The concept of Delaunay triangulations of discrete point sets in Euclidean spaces is well known. These triangulations are characterized by the property that the circumscribing sphere of each simplex does not contain any vertices in its interior; see Figure 5.11. In this section we will give the more general definition of a Delaunay triangulation of a metric space, we will show how this definition relates to the better known Euclidean case, and we will discuss uniqueness of these triangulations.

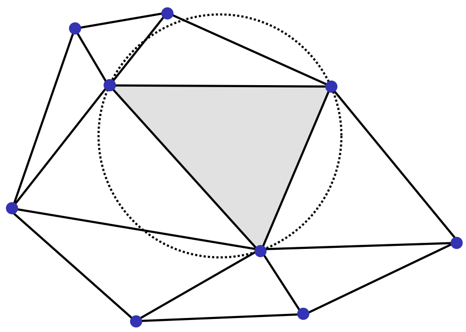

Figure 5.3: A planar Delaunay triangulation.

Let (X, d) be a metric space. A *minimal circumscribing sphere* of a point set C in X is a geodesic ball $B_{\mathrm{d}}(x; R)$ with the property that $C \subset B_{\mathrm{d}}(x; R)$, and such that R is minimal over all radii of spheres that satisfy this property. A *Delaunay triangulation* of (X, d) with vertex set $S \subset X$, is a triangulation $(\mathcal{X}, h)$ of X with vertex set S, such that for any simplex $\sigma \in \mathcal{M}$, any minimal circumscribing sphere of $h(V(\sigma))$ in (X, d) contains no points of S in its interior.

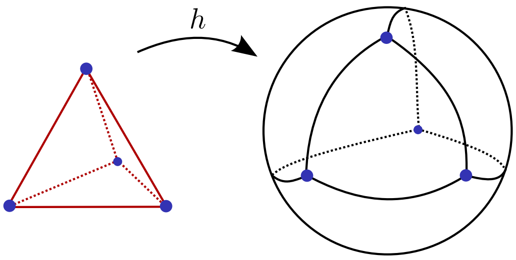

Figure 5.4: A triangulation of the sphere.

Relation to Euclidean Delaunay triangulations. Given a discrete set of points S in a Euclidean space, there exists a Delaunay triangulation $(\mathcal{D}, h)$ of $(\mathrm{hull}(S), \mathrm{d}_E)$ with vertex set S such that $h : |\mathcal{D}| \to \mathrm{hull}(S)$ is the identity map. The simplicial complex $\mathcal{D}$ of such a Delaunay triangulation is often called the Delaunay triangulation of S.

Uniqueness. Let M be a metric d-manifold with metric d. A point set $S \subset M$ is said to be in *general position* if no $d+2$ points of S lie on the boundary of some ball $B_{\mathrm{d}}(x; R)$. A Delaunay triangulation $(\mathcal{X}, h)$ of a metric space (X, d) with vertex set S is said to be *unique (up to isomorphism)* if all other Delaunay triangulations of (X, d) with vertex set S are isomorphic to $(\mathcal{X}, h)$. Just as in the Euclidean case, a Delaunay triangulation is unqiue if the vertices are in general position.

5.3 Delaunay Triangulations of Riemannian Manifolds

As we have seen in Section 2.4, a Riemannian structure of a manifold induces a metric on the manifold. This means that we can consider Delaunay triangulations of Riemannian manifolds. In this section we will use the results of Leibon and Letcher [11] to show that for each sufficiently dense net of a Riemannian manifold there exists a Delaunay triangulation of the manifold with this net as vertex set. We will start by discussion the concepts of strong convexity and density radius.

Let M be a Riemannian d-manifold with induced metric d. A subspace $C \subset M$ is said to be *strongly convex* if given any two points in C, there is a unique geodesic in C connecting them, and this geodesic is shorter than any other geodesic in M connecting these points. The *strong convexity radius* of a point $x \in M$ is the largest $R > 0$ such that $B_{\mathrm{d}}(x; R)$ is strongly convex. The *strong convexity radius* of the manifold M is the infimum of the strong convexity radii at all points in M. [11]

Lemma 5.1 ([11])**.** *The strong convexity radius of a compact Riemannian manifold is strictly positive.*

For any $x \in M$ define the *density radius*, $\mathrm{rad}(x)$, to be one-fifth of the strong convexity radius of $x \in M$. A set of points $S \subset M$ is said to satisfy the *density radius property* if for every $y \in M$ and $z \in B_{\mathrm{d}}(y; 4\,\mathrm{rad}(y))$ the ball of radius $\mathrm{rad}(y)$ centered at z contains a point of S in its interior. [11]

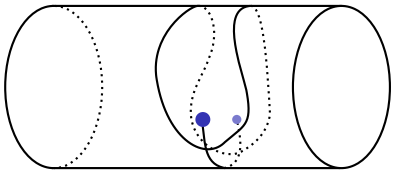

Figure 5.5: A region which is not strongly convex.

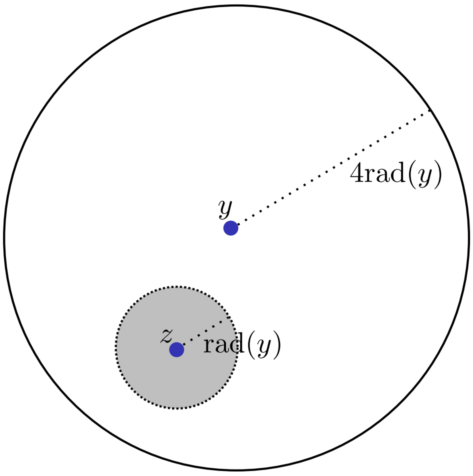

Figure 5.6: For S to satisfy the density radius property, the shaded region should contain a point of S.

Lemma 5.2 ([11])**.** *Let M be a compact Riemannian manifold with induced metric* d*. If S is a point set in M which is in general position and satisfies the density radius property in* (M, d)*, then there exists a unique Delaunay triangulation of* (M, d) *with vertex set S.*

The following lemma uses the fact that compact manifolds have strictly positive density radii to show that for $\epsilon > 0$ sufficiently small, there exist ϵ-nets which satisfy the density radius property. In the next corollary this will be used to show that for $\epsilon > 0$ sufficiently small there exists Delaunay triangulations of compact Riemannian manifolds with vertex set an ϵ-net.

Lemma 5.3. *Let M be a compact Riemannian manifold with induced metric* d*. For $\epsilon > 0$ sufficiently small, there exists a finite ϵ-net in* (M, d) *which is in general position and satisfies the density radius property.*

Proof. By Lemma 5.1, there is a $K > 0$ such that $\mathrm{rad}(x) \geq K$ for all $x \in M$. By Lemma 3.3 there exists a finite ϵ-net in the compact metric space (M, d) which is in general position, for each $0 < \epsilon \leq K$. Since S is in particular an ϵ-covering, every ball

$$B_{\mathrm{d}}(z; \epsilon) \subset B_{\mathrm{d}}(z; K) \subset B_{\mathrm{d}}(z; \mathrm{rad}(y))$$

contains a point of S, so S is satisfies the density radius property. □

Corollary 5.4. *Let M be a compact Riemannian manifold with induced metric* d*. For $\epsilon > 0$ sufficiently small, there exists a Delaunay triangulation* $(\mathcal{M}, h)$ *of* (M, d) *with vertex set an ϵ-net in* (M, d)*.*

An interesting property of Delaunay triangulations $(\mathcal{M}, h)$ where the vertex set is a net is that for any simplex $\sigma \in \mathcal{M}$, the points $h(V(\sigma))$ are guaranteed to be close in the metric in which the net is defined.

Lemma 5.5. *If $(\mathcal{M}, h)$ is a Delaunay triangulation of (M, d) with as vertex set an ϵ-net S in (M, d), then for any $\sigma \in \mathcal{M}$ there is a $p \in M$ such that $h(V(\sigma)) \subset B_{\mathrm{d}}(p; \epsilon)$.*

Proof. Since σ is a Delaunay simplex, there exists a ball $B_{\mathrm{d}}(p; R)$ such that the vertices of σ lie on the boundary of $B_{\mathrm{d}}(p; R)$ and the interior of $B_{\mathrm{d}}(p; R)$ does not contain a point of S. Since S is an ϵ covering we have $R \leq \epsilon$. $\square$

5.4 Meshing of Manifolds

A triangulation $(\mathcal{M}, h)$ of an embedded manifold M is called a *piecewise linear manifold mesh* if the restriction of h to $\mathcal{M}$ is the identity map, that is, $V(\mathcal{M}) \subset M$. In Figure 5.7 and 5.8 we depict the difference between a triangulation and a mesh. In this report we will refer to a piecewise linear manifold mesh simply by a *mesh*. Since a mesh is an approximation of a manifold it is also called a piecewise linear approximation.

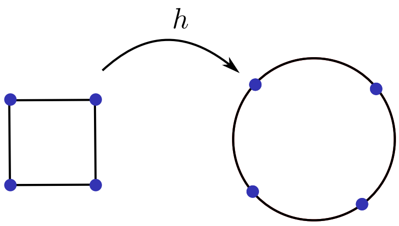

Figure 5.7: A triangulation.

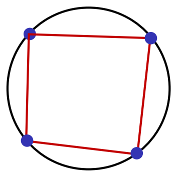

Figure 5.8: A mesh.

In Section 5.1 we noted that it is shown in [17, Chapter IV, Part B] that each smooth manifold admits a triangulation. The proof, however, shows that each smooth manifold in fact admits a mesh. The proof starts by noting that each smooth d-manifold can be embedded in $\mathbb{R}^{2d+1}$, using the Whitney embedding theorem [10, Theorem 6.12]. , and then constructs a simplicial complex whose vertices lie on M. The proof is completed by showing that the carrier of this simplicial complex is homeomorphic to M. This means, however, that the simplicial complex of the constructed triangulation is in fact a mesh. In this proof the d-manifold is embedded in $\mathbb{R}^{2d+1}$, however this assumption is there only as Whitney embedding theorem, and can be removed. That is, if a d-manifold is already embedded in a Euclidean space $\mathbb{R}^{d+c}$, then we can ship the step of embedding it in $\mathbb{R}^{2d+1}$, and proceed with the proof with the manifold embedded in $\mathbb{R}^{d+c}$. In conclusion, this proof shows that all smooth d-manifolds embedded in $\mathbb{R}^{d+c}$ admit a mesh.

In order to find a good upper bound on the Hausdorff distance between a manifold and a mesh we need certain condition on the vertices of the mesh. As we will see in Section 6.1, these condition are satisfied by an intrinsic Delaunay mesh. Given a compact Riemannian d-manifold M with induced metric d, which is embedded in $\mathbb{R}^{d+c}$, an *intrinsic Delaunay mesh* of M is a mesh of M which is also a Delaunay triangulation of M.

Intrinsic geodesic spheres. We use the term intrinsic in the definition of an intrinsic Delaunay triangulation to stress that the empty circumscribing spheres are geodesic spheres in (M, d), and are not, as one might expect when talking about meshes, Euclidean spheres. We will discuss the first fundamental form to make this more precise. Let M be a piecewise C^1 manifold, and let $\mathrm{d_I}$ be the metric given by

$$\mathrm{d_I}(p, q) = \inf_{\gamma \in \Gamma(p,q)} \int_0^1 \langle(\gamma'(t), \gamma'(t)\rangle \, \mathrm{d}t,$$

where $\langle ., . \rangle$ is the Euclidean inner product on $\mathbb{R}^{d+c}$. Note that when M is of class C^1 and q_I is the Riemannian tensor field on M induced by the inner product of the ambient space, then $\mathrm{d_I}$ is just the metric induced by q_I.

If $\mathcal{M}$ is an intrinsic Delaunay triangulation of a compact Riemannian d-manifold M with induced metric d, which is embedded in $\mathbb{R}^{d+c}$, then we have an empty sphere condition in (M, d), and we do not have an empty sphere condition in $(\mathbb{R}^{d+c}, \mathrm{d}_E)$, (M, d_I), or $(|\mathcal{M}|, \mathrm{d_I})$.

In [4, Section 4.1] it is even shown that there exists a smooth d-manifold M embedded in $\mathbb{R}^{d+c}$, such that there exists arbitrary dense intrinsic Delaunay meshes $\mathcal{M}$ of $(M, \mathrm{d_I})$ which contain simplices which do not satisfy the empty sphere condition in $(|\mathcal{M}|, \mathrm{d_I})$.

Ambient isotopies. Two topological spaces are often regarded to be topologically equivalent if they are homeomorphic. However, if we embed these manifolds in a Euclidean space, then they can have quite different embeddings while they are homeomorphic; see Figure 5.9. This is one of the reasons that we sometimes require embedded manifolds to be ambiently isotopic to regard them as being topologically equivalent. Given subspaces $A, B \subset \mathbb{R}^n$, an *ambient isotopy* from A to B is a continuous map $h : \mathbb{R}^n \times [0, 1] \to \mathbb{R}^n$ such that $h(., t) : \mathbb{R}^n \to \mathbb{R}^n$ is a homeomorphism for each $t \in [0, 1]$, $h(., 0) = \mathrm{Id}_{\mathbb{R}^n}$, and $h(A, 1) = B$. If two spaces are ambiently isotopic, then they are in particular homeomorphic.

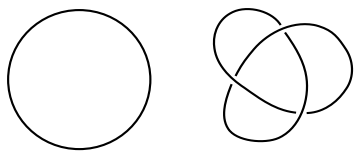

Figure 5.9: Topological equivalence.

5.5 From a Triangulation to a Mesh

The *linear approximation* of a triangulation $(\mathcal{T}, h)$ of an embedded manifold M is defined by

$$E_h : \sigma \in \mathcal{T} \mapsto \mathrm{hull}(h(V(\sigma))).$$

Recall that we defined a triangulation to be a simplicial complex together with a homeomorphism from its carrier to the manifold, and a mesh to be a triangulation where this homeomorphism restricted to the vertex set is the identity. This means that we can use a linear approximation of a triangulation of a manifold to obtain a mesh of the manifold.

There can be both local and global obstructions for a linear approximation $E_h(\mathcal{T})$ to be a simplicial complex, but if $E_h(\mathcal{T})$ can be shown to be a simplicial complex, then it follows immediately from the definitions that $E_h(\mathcal{T})$ is a mesh of M. Furthermore, if $(\mathcal{T}, h)$ is a Delaunay triangulation of M and the linear approximation $E_h(\mathcal{T})$ is a simplicial complex, then it follows immediately that $E_h(\mathcal{T})$ is an intrinsic Delaunay mesh of M.

We will show that if the triangulation is a Delaunay triangulation with as vertex set a sufficiently dense net, then these obstructions will not occur, which means that the linear approximation is a well defined simplicial complex, and hence is an intrinsic Delaunay mesh.

5.5.1 Local Discussion

Two simplices which share a face are called *simplicial neighbours*. If the intersection of two simplicial neighbours in a collection of simplices $\mathcal{M}$ does not solely consist of their shared face, then $\mathcal{M}$ is not a simplicial complex. This situation can occur for a collection of simplices which is the linear approximation of a triangulation, as depicted in Figure 5.10.

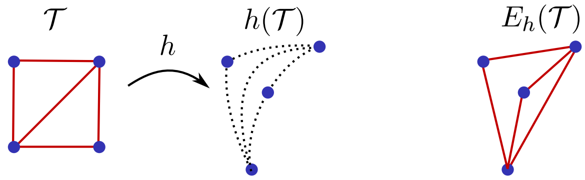

Figure 5.10: A linear approximation which is not a mesh.

To show that a linear approximation does not contain such local obstructions, we will show that its local subcollections are well defined simplicial complexes. Given a differentiable d-manifold M embedded in $\mathbb{R}^{d+c}$, we define the *local subcollection* of $\mathcal{M}$ at $p \in M$ by

$$\operatorname{lsc}_p(\mathcal{M}; h) = \{\sigma \in \mathcal{M} \mid h(V(\sigma)) \subset U_p\}.$$

The neighbourhood U_p which we use in this definition is defined in Section 4.2. Note that if $(\mathcal{T}, h)$ is a triangulation of M and $p \in M$, then by using the notation $\operatorname{lsc}_p(.) = \operatorname{lsc}_p(.; \mathrm{Id})$, we have that

$$E_h(\operatorname{lsc}_p(\mathcal{T}; h)) = \operatorname{lsc}_p(E_h(\mathcal{T})).$$

We use the neighbourhood U_p in the definition of a local subcollection to be able to project the simplices injectively on the tangent space, and we will show that a local subcollection is a simplicial complex by showing that its projection is a simplicial complex. For this we will need the following two lemmas.

Lemma 5.6. *If ϕ is a diffeomorphism of class C^1 whose partial derivatives are locally Lipschitz continuous, whose domain includes a neighbourhood of $0 \in \mathbb{R}^d$, where $d \geq 1$, and whose codomain is $\mathbb{R}^d$, then there exists an $R > 0$ such that $\phi(B_{\mathrm{d}_E}(0; r))$ is convex for all $0 < r < R$.*

The case for $d = 1$ is trivial. Below we only give a proof for $d = 2$; see Section 8.2 in the Future Work chapter.

Proof. Since the entries of $D\phi$ are locally Lipschitz continuous, there is a $K > 0$ such that for all $R > 0$ sufficiently small,

$$\mathrm{d}_E((D\phi)_{ij}(r\nu_1), (D\phi)_{ij}(r\nu_2)) \leq K \mathrm{d}_E(r\nu_1, r\nu_2) < RK \mathrm{d}_E(\nu_1, \nu_2)$$

for all $\nu_1, \nu_2 \in \mathbb{S}^1$, all $0 < r < R$, and all $1 \leq i, j \leq n$. If we denote by $t(r\nu)$ the counter clockwise unit tangent vector in $T_{r\nu} S_{\mathrm{d}_E}(0; r)$, then by taking R small, the variation in the entries of $\nu \mapsto D\phi(r\nu)$ can be made arbitrarily small with respect to the variation in the entries of $\nu \mapsto t(r\nu)$. This means that

$$\operatorname{normalize}(D\phi(r\nu_1)t(r\nu_1)) \neq \operatorname{normalize}(D\phi(r\nu_2)t(r\nu_2))$$

for all $\nu_1, \nu_2 \in \mathbb{S}^1$ with $\nu_1 \neq \nu_2$ and all $0 < r < R$. Therefore there exists an $R > 0$ such that

$$\psi : \mathbb{S}^1 \to \mathbb{S}^1,\ \nu \mapsto \text{normalize}(D\phi(r\nu)t(r\nu))$$

is injective for all $0 < r < R$.

Since the derivative $D\phi(r\nu_1)$ at $r\nu_1$ maps $T_{r\nu}S_{\mathrm{d}_E}(0;r)$ to $T_{\phi(r\nu)}\phi(S_{\mathrm{d}_E}(0;r))$, the map ψ provides a global injective unit tangent vector field of $\phi(S_{\mathrm{d}_E}(0;r))$. So rotating these vectors by 90 degrees yields an injective normal vector field of

$$\phi(S_{\mathrm{d}_E}(0;r)) = \phi(\partial B_{\mathrm{d}_E}(0;r)) = \partial\phi(B_{\mathrm{d}_E}(0;r)),$$

which implies that $\phi(B_{\mathrm{d}_E}(0;r))$ is convex. □

Note that the class of C^1 diffeomorphisms with locally Lipschitz continuous partial derivatives contains in particular the class of C^2 diffeomorphisms and the class of $C^{1,1}$ diffeomorphisms. We will now use this result to show that the projection of small enough geodesic balls is convex. Besides using the result of the following lemma in this section, this result will also be used in the proof of Lemma 6.1.

Lemma 5.7. *Let M be a compact Riemannian d-manifold of class C^2 with induced metric* d*, and let M be embedded in $\mathbb{R}^{d+c}$, where $d, c \geq 1$. For each $p \in M$ let η_p be some orthonormal tangent frame at p. There exists an $R > 0$ such that for each $p \in M$, each $q \in U_p$, and each $0 < r < R$ for which $B_\mathrm{d}(q;r) \subset U_p$, the projected ball $\tau_{\eta_p}(B_\mathrm{d}(q;r))$ is convex.*

Proof. Let $p \in M$ and $q \in U_p$. Let $\exp_q : V_q \subset T_qM \to M$ be the exponential map at q and let $E : \mathbb{R}^n \to T_qM$ be an isomorphism induced by the choice of an orthonormal basis on T_qM. In this way we have that for r small enough $B_\mathrm{d}(q;r) = (\exp_q \circ E)(B_{\mathrm{d}_E}(0;r))$. We also have that $\exp_q \circ E$ is a C^2 diffeomorphism since the Riemannian structure is of class C^2.

Since the manifold is of class C^2 we also have that the projection map $\tau_{\eta_p} : U_p \to \mathbb{R}^d$ is a C^2 diffeomorphism. So, the composition $\tau_{\eta_p} \circ \exp_q \circ E : \mathbb{R}^n \to \mathbb{R}^n$ is of class C^2 which means that we can use Lemma 5.6 to conclude that

$$\tau_{\eta_p}(B_\mathrm{d}(q;r)) = (\tau_{\eta_p} \circ \exp_q \circ E)(B_{\mathrm{d}_E}(0;r))$$

is convex for r sufficiently small. We denote the supremum over all radii r for which $\tau_{\eta_p}(B_\mathrm{d}(q;r))$ is convex by R_p. By the above result and by compactness of M, we have that $0 < R_p < \infty$ for all $p \in M$.

The map $p \mapsto R_p$ is continuous since the Riemannian structure of M is of class C^2. Since in addition M is compact, if follows from the extreme value theorem that $R = \min_{p \in M} R_p > 0$ is defined, which completes the proof. □

We use that the projections of small geodesic balls are convex to prove that the local subcollections of the linear approximations of certain triangulations are well defined simplicial complexes.

Lemma 5.8. *Let M be a compact Riemannian d-manifold of class C^2 with induced metric* d*, and let M be embedded in $\mathbb{R}^{d+c}$, with $1 \leq d \leq 2$ and $c \geq 1$. Let $(\mathcal{T}, h)$ be a Delaunay triangulation of (M, d) with as vertex set an ϵ-net in (M, d). If $\epsilon > 0$ is sufficiently small, then*

$$lsc_p(E_h(\mathcal{T}))$$

is a simplicial complex for each $p \in M$.

Proof. Since the proof is trivial for the case where $d = 1$ we will assume that $d = 2$. Since h is in particular a bijection on the vertices we have that $\sigma_1 \in \mathrm{lsc}_p(\mathcal{T}, h)$ is a face of $\sigma_2 \in \mathrm{lsc}_p(\mathcal{T}, h)$ if and only if $E_h(\sigma_1) \in \mathrm{lsc}_p(E_h(\mathcal{T}))$ is a face of $E_h(\sigma_2) \in \mathrm{lsc}_p(E_h(\mathcal{T}))$. Furthermore, from the definition of E_h it follows that for all $\sigma_1, \sigma_2 \in \mathrm{lsc}_p(\mathcal{T}, h)$,

$$E_h(\sigma_1 \cap \sigma_2) \subset E_h(\sigma_1) \cap E_h(\sigma_2).$$

So, to show that $\mathrm{lsc}_p(E_h(\mathcal{T}))$ is a simplicial complex, we need to show that for all simplices $\sigma_1, \sigma_2 \in \mathrm{lsc}_p(\mathcal{T}, h)$ the intersection $E_h(\sigma_1) \cap E_h(\sigma_2)$ is contained in $E_h(\sigma_1 \cap \sigma_2)$.

To show this we will use projections of the simplices. Given a simplex $\sigma \in \mathrm{lsc}_p(\mathcal{T}, h)$, let

$$\phi(\sigma) = \pi_{\eta_p} E_h(\sigma)$$

be the projection of the linear approximation and let

$$\psi(\sigma) = \tau_{\eta_p} h(\sigma)$$

be the projection of the embedded simplex, where η_p is an orthonormal tangent frame at p.

For a simplex $\sigma \in \mathrm{lsc}_p(\mathcal{T}, h)$, the vertices of $\phi(\sigma)$ are the same as the vertices of $\psi(\sigma)$, and for small ϵ, the edges of $\psi(\sigma)$ are almost straight line segments. In the linear approximation these edges will be linearized, and if in this process the edges do not cross vertices, then it follows from the fact that $\psi(\mathrm{lsc}_p(\mathcal{T}, h))$ is a well defined embedded simplicial complex that

$$\phi(\sigma_1) \cap \phi(\sigma_2) \subset \phi(\sigma_1 \cap \sigma_2),$$

for all $\sigma_1, \sigma_2 \in \mathrm{lsc}_p(\mathcal{T}, h)$.

More precisely, we will show that for a 1-face σ_f of a 2-simplex $\sigma \in \mathrm{lsc}_p(\mathcal{T}, h)$, the closed curve consisting of the segments $\phi(\sigma_f)$ and $\psi(\sigma_f)$ does not bound any vertices. Since $(\mathcal{T}, h)$ is a Delaunay triangulation with as vertex set an ϵ-net, by Lemma 5.5 we have that there exists a geodesic disk D in (M, d) with radius at most ϵ such that the vertices of $h(V(\sigma_f))$ lie on the boundary of D, and the interior of D contains no vertices. For ϵ sufficiently small, we have by Lemma 5.7 that the projection $\tau_{\eta_p}(D)$ is convex, which means that $\phi(\sigma) \subset \tau_{\eta_p}(D)$. We also have that for ϵ sufficiently small, D is strongly convex, which means that $h(\sigma_f) \subset D$, which in turn implies that $\psi(\sigma) \subset \tau_{\eta_p}(D)$. So we have that both $\phi(\sigma_f)$ and $\psi(\sigma_f)$ are contained in $\tau_{\eta_p}(D)$, and since $\tau_{\eta_p}(D)$ contains no vertices other than the endpoints $V(\phi(\sigma_f)) = V(\psi(\sigma_f))$ we know that closed curve given by the segments $\phi(\sigma_f)$ and $\psi(\sigma_f)$ does not contain any vertices. $\square$

5.5.2 Global Discussion

We have seen sufficient conditions on a triangulation $(\mathcal{T}, h)$ for the local subcollections $\mathrm{lsc}_p(E_h(\mathcal{T}))$ of the linear approximation $E_h(\mathcal{T})$ to be simplicial complexes. This does not imply, however, that under these conditions the full collection $E_h(\mathcal{T})$ is a simplicial complex. As depicted in Figure 5.11 there can also be global obstructions for the linear approximation to be a simplicial complex. We will use the concepts of the medial axis and local feature size to show that these obstruction can be resolved.

The *medial axis* of an embedded manifold $M \subset \mathbb{R}^{d+c}$ is the closure of the set of points in $\mathbb{R}^{d+c}$ with more than one nearest neighbour in M. Since each point of M has only itself as nearest neighbour, the manifold and its medial axis are disjoint. The *local feature size*, $\mathrm{lfs}(x)$,

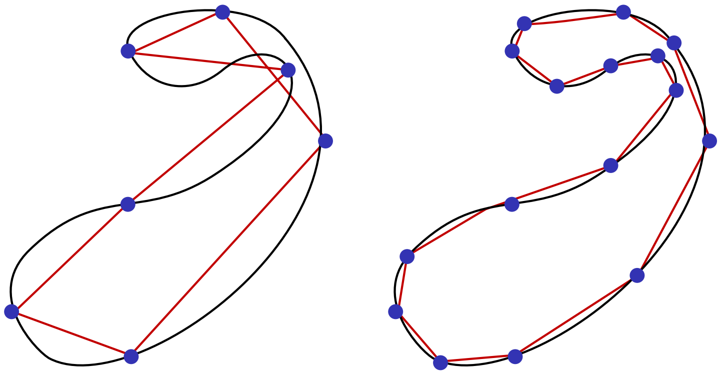

Figure 5.11: A linear approximation which is not a mesh due to a global obstruction.

of a point $x \in M$ is defined as the distance between x and the medial axis of M. This is well defined, since the medial axis is closed. An embedded manifold is said to be of *positive reach* if its *reach*

$$\mathrm{reach}(M) = \inf_{x \in M} \mathrm{lfs}(x)$$

is strictly greater than zero.

Lemma 5.9. *A compact manifold M embedded in a Euclidean space is of positive reach.*

Proof. The function $\mathrm{lfs} : M \to \mathbb{R}_{>0}$ is continuous, so by an application of the Weierstrass extreme value theorem $\mathrm{reach}(M)$ is equal to $\min_{x \in M} \mathrm{lfs}(x)$. Since M is disjoint form its medial axis $\mathrm{lfs}(x) > 0$ for all $x \in M$, so $\mathrm{reach}(M) > 0$. $\square$

We use that a manifold is of positive reach to show that each simplex is contained in a Euclidean ball with radius the reach of the manifold.

Lemma 5.10. *Let M be a compact Riemannian d-manifold of class C^2 with induced metric* d*, and let M be embedded in $\mathbb{R}^{d+c}$, where $d, c \geq 1$. Let $(\mathcal{T}, h)$ be a Delaunay triangulation of (M, d) with as vertex set an ϵ-net in (M, d). If $\epsilon > 0$ is sufficiently small, then for each simplex $\sigma \in E_h(\mathcal{T})$ there is a $p \in M$ such that $\sigma \subset B_{\mathrm{d}_E}(p; reach(M))$.*

Proof. By Lemma 5.5, for all $\sigma \in \mathcal{T}$ there is a $p \in M$ such that $h(V(\sigma)) \subset B_{\mathrm{d}}(p; \epsilon)$. Since $\mathcal{T}$ and $E_h(\mathcal{T})$ have the same combinatorial structure and $h(V(\mathcal{T})) = V(E_h(\mathcal{T}))$, we have that for all $\sigma \in E_h(\mathcal{T})$ there is a $p \in M$ such that $V(\sigma) \subset B_{\mathrm{d}}(p; \epsilon)$.

Since the metric d and the restriction of the Euclidean metric d_E to M induce the same topology on M, and since M is compact, we have that for ϵ sufficiently small,

$$B_{\mathrm{d}}(p; \epsilon) \subset B_{\mathrm{d}_E}(p; \mathrm{reach}(M))$$

for all $p \in M$. So, for each $\sigma \in E_h(\mathcal{T})$ there is a $p \in M$ such that

$$V(\sigma) \subset B_{\mathrm{d}}(p; \epsilon) \subset B_{\mathrm{d}_E}(p; \mathrm{reach}(p)). \qquad \square$$

We can use the previous result to show that simplices which lie far apart, as measured along the manifold, do not intersect.

Lemma 5.11. *Let M be a compact Riemannian d-manifold of class C^2 with induced metric* d*, and let M be embedded in $\mathbb{R}^{d+c}$, where $d, c \geq 1$. Let $(\mathcal{T}, h)$ be a Delaunay triangulation of (M, d) with as vertex set an ϵ-net in (M, d). If $\epsilon > 0$ is sufficiently small and σ_1 and σ_2 are simplices in $E_h(\mathcal{T})$ such that there is no $p \in M$ for which both σ_1 and σ_2 lie in $lsc_p(E_h(\mathcal{T}))$, then $\sigma_1 \cap \sigma_2 = \emptyset$.*

Proof. Let $\sigma_1, \sigma_2 \in E_h(\mathcal{T})$ such that there is no $p \in M$ for which $\sigma_1, \sigma_2 \in \mathrm{lsc}_p(E_h(\mathcal{T}))$. This means that the intersection of $\phi(\sigma_1)$ and $\phi(\sigma_2)$ is empty for ϵ sufficiently small.

By Lemma 5.10 there exist $p_1, p_2 \in M$ such that $\sigma_i \subset B_{\mathrm{d}_E}(p_i; \mathrm{reach}(M))$. So the restriction of ϕ to $\sigma_1 \cup \sigma_2$ is a bijection, so the intersection of σ_1 and σ_2 is empty. □

We can now combine the previous result with the main result of the previous section to obtain that the linear approximations of certain triangulations are simplicial complexes.

Theorem 5.12. *Let M be a compact Riemannian d-manifold of class C^2 with induced metric* d*, and let M be embedded in $\mathbb{R}^{d+c}$, where $1 \leq d \leq 2$ and $c \geq 1$. If $(\mathcal{T}, h)$ is a Delaunay triangulation of (M, d) with as vertex set an ϵ-net in (M, d) with $\epsilon > 0$ sufficiently small, then the linear approximation $E_h(\mathcal{T})$ is a simplicial complex.*

Proof. We will start by showing that $E_h(\mathcal{T})$ is a simplicial complex. Since $E_h(\mathcal{T})$ is a linear approximation of a simplicial complex, it contains all faces of all simplices contained in it. Let σ_1 and σ_2 be two simplices in $E_h(\mathcal{T})$. If there is a $p \in M$ such that $\sigma_1, \sigma_2 \in \mathrm{lsc}_p(E_h(\mathcal{T}))$, then, by Lemma 5.8, the intersection of σ_1 and σ_2 is either empty or a shared face. If there does not exists a $p \in M$ such that $\sigma_1, \sigma_2 \in \mathrm{lsc}_p(E_h(\mathcal{T}))$, then, by Lemma 5.10, the intersection of σ_1 and σ_2 is empty. So $E_h(\mathcal{T})$ is a simplicial complex. □

5.5.3 Topological Equivalence

As noted in the introduction of this section, now that we know that the linear approximation is a simplicial complex, it is not hard to show that it is homeomorphic to the manifold.

Theorem 5.13. *Let M be a compact Riemannian d-manifold of class C^2 with induced metric* d*, and let M be embedded in $\mathbb{R}^{d+c}$, where $1 \leq d \leq 2$ and $c \geq 2$. If $(\mathcal{T}, h)$ is a Delaunay triangulation of (M, d) with as vertex set an ϵ-net in (M, d) with $\epsilon > 0$ sufficiently small, then the linear approximation $E_h(\mathcal{T})$ is an intrinsic Delaunay mesh of M.*

Proof. Since $E_h(\mathcal{T})$ is an linear approximation of $\mathcal{T}$ which is also a simplicial complex, there exists a homeomorphism

$$g : |E_h(\mathcal{T})| \to |\mathcal{T}|.$$

The composition $h \circ g$ is a homeomorphism from $|E_h(\mathcal{T})|$ to M, so $(|E_h(\mathcal{T})|, h \circ g)$ is a triangulation of M. The triangulation $(|E_h(\mathcal{T})|, h \circ g)$ has the same vertex set as $(\mathcal{T}, h)$, so the former is also a Delaunay triangulation, and since $h \circ g$ restricted to $V(E_h(\mathcal{T}))$ is the identity, $E_h(\mathcal{T})$ is an intrinsic Delaunay mesh of M □

In Section 5.4 we discussed the possibility of having the additional requirement that the carrier of a mesh should be ambiently isotopic to the manifold. To show that this is possible we need to discuss tubular neighbourhoods and its fibres as discussed in [2]. Recall from Section 2.3 that a *tubular neighbourhood* of a subset $S \subset \mathbb{R}^{d+c}$ is given by

$$\{x \in \mathbb{R}^{d+c} \mid \mathrm{d}(x, S) \leq R\}.$$

Given a smooth compact d-manifold M embedded in $\mathbb{R}^{d+c}$, let T_M be the tubular neighbourhood of M with radius $\mathrm{reach}(M)$. Since for each point in T_M there is by definition only one nearest point on M, we have a well defined map $\pi_M : T_M \to M$, which maps each point to the nearest point on M. The inverse images $\pi_M^{-1}(\{p\})$ where $p \in M$ are called the *fibres* of T_M. If a subset of T_M intersects with each fibre in precisely one point, then the restriction of π_M to this subset yields a bijection. The following lemma uses this idea to construct an ambient isotopy.

Lemma 5.14 ([13, Theorem 4.1])**.** *If M is a compact 2-manifold of class C^2 embedded in $\mathbb{R}^3$, and W is a compact 2-manifold which is contained in a tubular neighbourhood T of M such that each fibre of T intersects W in precisely one point, then π_T induces an ambient isotopy from M to W.*

It follows from the proof of Lemma 5.8 that for an intrinsic Delaunay mesh where the vertex set is a sufficiently dense net, the condition that each fibre is intersected in precisely one point is satisfied. So we obtain the following corollary, which can be generalized to higher dimension and codimension by generalizing Lemma 5.14.

Corollary 5.15. *Let M be a compact Riemannian 2-manifold with induced metric d, and let M be embedded in $\mathbb{R}^3$. If $(\mathcal{T}, h)$ is a Delaunay triangulation of (M, d) with as vertex set an ϵ-net in (M, d) with $\epsilon > 0$ sufficiently small, then the carrier the linear approximation $E_h(\mathcal{T})$ is ambiently isotopic to M.*

Chapter 6

Upper Bounds on the Approximation Error

In this chapter we will combine the results of the previous chapters to obtain asymptotic upper bounds on the Hausdorff distance between manifolds and the carriers of optimal meshes. These bounds are functions of the number of vertices, and by an optimal mesh we mean that given the number of vertices, the Hausdorff distance in minimized. After having established these upper bounds in Section 6.1, we will discuss special cases in Section 6.2. For some of these special cases results are already known, so this allows us to compare the results. In Section 6.3 we discuss some examples to illustrate the computation of upper bounds using the results of Theorem 6.3.

6.1 Upper Bounds

To construct upper bounds we start by defining curvature tensor fields on the manifolds as defined in Chapter 4. Then we construct ϵ-nets, as defined in Chapter 3, in the metric space induced by these curvature tensor fields. We will use these nets as the vertex sets of intrinsic Delaunay meshes, as defined in Chapter 5. In Lemma 6.1 we will bound the approximation error in ϵ, and in Lemma 6.2 we will use that ϵ can be bounded in the number of vertices to bound the approximation error in the number of vertices.

To bound the approximation error in ϵ we use that the approximating mesh is in particular a Delaunay triangulation, and its vertex set an ϵ-covering. This implies that for small ϵ, the vertices of a simplex of a mesh are close to each other with respect to the curvature of the manifold. We use the result of Lemma 5.7 to obtain a good constant in the asymptotic formula.

Lemma 6.1. *Let M be a compact differentiable d-manifold of class C^2 embedded in $\mathbb{R}^{d+c}$, with $d, c \geq 1$, which admits a nonoriented orthonormal normal frame field ν. If for each $\delta > 0$ and each sufficiently small $\epsilon > 0$, $\mathcal{M}_\epsilon^\delta$ is an intrinsic Delaunay mesh of (M, d_ν^δ) with as vertex set an ϵ-net in (M, d_ν^δ), then*

$$\mathrm{d}_H(M, |\mathcal{M}_\epsilon^\delta|) \lesssim 2\epsilon^2 \quad as \quad \epsilon \downarrow 0 \quad and \quad \delta \downarrow 0.$$

Proof. For each $q \in M$ we denote by η_q an arbitrary orthonormal tangent frame at q. As we have seen in Section 5.5.3, for $\epsilon > 0$ sufficiently small, the restriction of the map $\pi_M : T_M \to M$

to $|\mathcal{M}_\epsilon^\delta|$ is a homeomorphism $\hat{\pi}_M : |\mathcal{M}_\epsilon^\delta| \to M$ which projects the points of $|\mathcal{M}_\epsilon^\delta|$ down along the fibres of the tubular neighbourhood T_M. That is, for each $t \in |\mathcal{M}_\epsilon^\delta|$, $r = \hat{\pi}_M(t)$ is the point such that $\pi_{\eta_r} t = 0$.

In this proof we will show that for arbitrary $\lambda > 1$ there exists scalars $\hat{\epsilon}, \hat{\delta} > 0$ such that for each $\epsilon \in (0, \hat{\epsilon}]$, each $\delta \in (0, \hat{\delta}]$, and each pair $(t, \pi_M(t))$ where $t \in M$,

$$\mathrm{d}_E(t, \hat{\pi}_M(t)) \leq 2\lambda\epsilon^2.$$

Since π_M is a bijection and since $\hat{\epsilon}$ and $\hat{\delta}$ do not depend on the pair $(t, \hat{\pi}_M(t))$, this means that

$$\mathrm{d}_H(M, |\mathcal{M}_\epsilon^\delta|) \lesssim 2\epsilon^2 \quad \text{as} \quad \epsilon \downarrow 0 \quad \text{and} \quad \delta \downarrow 0.$$

Let $\lambda > 1$. We start by defining the scalars $\hat{\epsilon}$ and $\hat{\delta}$. There is a $\hat{\delta} > 0$ such that for all $0 < \delta < \hat{\delta}$ and all $r \in M$, there is an open neighbourhood $V_r \subset U_r$ of r such that we can apply Corollary 4.15 with the neighbourhood V_r and the scalars $\hat{\delta}$ and λ. The collection $\{V_r \mid r \in M\}$ is an open cover of the compact metric space $(M, \mathrm{d}_\nu^\delta)$. So, by the Lebesgue number lemma [12, Lemma 7.2], there is an $\hat{\epsilon}_1 > 0$ such that for all $0 < \epsilon < \hat{\epsilon}_1$ and all $r \in M$, the ball $B_{\mathrm{d}_\nu^\delta}(r; 2\epsilon)$ is contained in V_r.

By Lemma 5.7, there is an $\hat{\epsilon}_2$ such that if $0 < \epsilon \leq \hat{\epsilon}_2$, then for each $p \in M$ for which $B_{\mathrm{d}_\nu^\delta}(p; \epsilon) \subset U_r$, the projection $\tau_{\eta_r}(B_{\mathrm{d}_\nu^\delta}(p; \epsilon))$ is convex. Let $\hat{\epsilon} = \min\{\hat{\epsilon}_1, \hat{\epsilon}_2, \hat{\epsilon}_3\}$, where $\hat{\epsilon}_3 > 0$ is small with respect to the reach of M.

Let $\epsilon \in (0, \hat{\epsilon}]$ and $\delta \in (0, \hat{\delta}]$. Let r be an arbitrary element in M. Let t be the point in $|\mathcal{M}_\epsilon^\delta|$ such that $\pi_M(t) = r$, that is, $\pi_{\eta_r} t = 0$. Let σ be the simplex in $\mathcal{M}_\epsilon^\delta$ such that $t \in \sigma$.

By Lemma 5.5 there exists a $p \in M$ such that $V(\sigma) \subset B_{\mathrm{d}_\nu^\delta}(p; \epsilon)$. We have chosen $\hat{\epsilon}_3$ such that $B_{\mathrm{d}_\nu^\delta}(p; \epsilon) \subset U_r$. By the choice of $\hat{\epsilon}_2$ this means that $\tau_{\eta_r}(B_{\mathrm{d}_\nu^\delta}(p; \epsilon))$ is convex, so $\pi_{\eta_r}(\sigma) \subset \tau_{\eta_r}(B_{\mathrm{d}_\nu^\delta}(p; \epsilon))$. Since $\pi_{\eta_r} t = 0$ and $t \in \sigma$, this means that $0 \in \tau_{\eta_r}(B_{\mathrm{d}_\nu^\delta}(p; \epsilon))$. This means that $r \in B_{\mathrm{d}_\nu^\delta}(p; \epsilon)$, which implies that $\mathrm{d}_\nu^\delta(v, r) \leq 2\epsilon$ for all $v \in V(\sigma)$. This also implies that

$$B_{\mathrm{d}_\nu^\delta}(p; \epsilon) \subset B_{\mathrm{d}_\nu^\delta}(r; 2\epsilon) \subset V_r,$$

which means that we can apply Corollary 4.15.

So for $v \in V(\sigma)$, we have

$$\mathrm{d}_E(v, T_r M) \leq \frac{\lambda}{2} \mathrm{d}_\nu^\delta(v, r)^2 \leq 2\lambda\epsilon^2,$$

where the first inequality follows from Corollary 4.15. The point t is a convex combination of the vertices of σ, so

$$\mathrm{d}_E(t, r) = \mathrm{d}_E(t, T_r M) \leq \max_{v \in V(\sigma)} \mathrm{d}_E(v, T_r M) \leq 2\lambda\epsilon^2. \qquad \square$$

In the proof of the following lemma we use that ϵ can be bounded in the number of vertices of the mesh by using that the vertex set is an ϵ-packing. We then combine this with the result of the previous lemma to obtain an upper bound of the error in the number of vertices.

Lemma 6.2. *Let M be a compact differentiable d-manifold embedded in $\mathbb{R}^{d+c}$, with $d, c \geq 1$, which admits a nonoriented orthonormal normal frame field ν. If for each $\delta > 0$ and each sufficiently large n, $\mathcal{M}_n^\delta$ is an intrinsic Delaunay mesh of (M, d_ν^δ) with as vertex set S_n a net of cardinality n in (M, d_ν^δ), then*

$$\mathrm{d}_H(M, |\mathcal{M}_n^\delta|) \lesssim 8 \left(\frac{\lambda_d \mu_\nu(M)}{n V_d} \right)^{2/d} \quad \text{as} \quad n \to \infty \quad \text{and} \quad \delta \downarrow 0.$$

Proof. The net S_n is in particular an ϵ_n-packing for some ϵ_n, so, by Lemma 3.10, we have that for all $\delta > 0$,

$$\epsilon_n \lesssim 2 \left(\frac{\lambda_d \mu_\nu^\delta(M)}{nV_d} \right)^{1/d} \quad \text{as} \quad n \to \infty.$$

By Lemma 6.1 we have that

$$\mathrm{d}_H(M, |\mathcal{M}_n^\delta|) \lesssim 2\epsilon_n^2 \quad \text{as} \quad n \to \infty \quad \text{and} \quad \delta \downarrow 0.$$

This implies that

$$\mathrm{d}_H(M, |\mathcal{M}_n^\delta|) = \max\{\mathrm{d}_E(M, |\mathcal{M}_n^\delta|), \mathrm{d}_E(|\mathcal{M}_n^\delta|, M)\} \lesssim 2\epsilon_n^2 \lesssim 8 \left(\frac{\lambda_d \mu_\nu(M)}{nV_d} \right)^{2/d},$$

as $n \to \infty$ and $\delta \downarrow 0$, which completes the proof. □

In the previous two lemmas we require a collection of meshes to be given. By using results from Chapter 5 we can remove this condition, that is, we will show that there always exists a collection of meshes which satisfies the given asymptotic upper bound. This theorem is a generalization to higher codimension of Theorem 4.2 of Clarkson [3]. By using the previous lemmas we are able to give an explicit expression for the constant in the asymptotic notation, and in Section 6.2 we will see that this upper bound is similar to already known results in special cases. Since we do only have a proof of Lemma 5.8 in the 1 or 2 dimensional case, we also have this restriction in the following theorem.

Theorem 6.3. *Let M be a compact differentiable d-manifold of class C^2 embedded in $\mathbb{R}^{d+c}$, where $1 \leq d \leq 2$ and $c \geq 1$, which admits a nonoriented orthonormal normal frame field ν. For each $\delta > 0$ and each sufficiently large n, there exists an intrinsic Delaunay mesh $(\mathcal{M}_n^\delta, h)$ of $(M, \mathrm{d}_\nu^\delta)$ with as vertex set a net in $(M, \mathrm{d}_\nu^\delta)$ of cardinality n, such that*

$$\mathrm{d}_H(M, |\mathcal{M}_n^\delta|) \lesssim 8 \left(\frac{\lambda_d \mu_\nu(M)}{nV_d} \right)^{2/d} \quad \textit{as} \quad n \to \infty \quad \textit{and} \quad \delta \downarrow 0.$$

Proof. By Corollary 5.4 and Lemma 3.10 we have that for sufficiently large n, there exists a Delaunay triangulation $(\mathcal{M}, h)$ of (M, d) with as vertex set a net of cardinality n in (M, d). By Lemma 5.13 it follows that the linear approximation $E_h(\mathcal{M})$ is a intrinsic Delaunay mesh of (M, d) with vertex set a net of cardinality n in (M, d). It follows from 6.1 that for each sufficiently large n, there is an $\epsilon > 0$ such that $E_h(\mathcal{M})$ is an intrinsic Delaunay mesh of (M, d) with vertex set an net of cardinality n in (M, d). So the proof is completed by an application of Lemma 6.2. □

This theorem shows that there exist meshes which admit the given asymptotic upper bound on the approximation error, and this means that this theorem gives an asymptotic upper bound on the optimal approximation error of a manifold. As we will see in Chapter 6.2, in some special cases where the optimal asymptotic approximation error is known, this upper bound is a factor 16 too large. A reason for this is that an intrinsic Delaunay mesh with as vertex set a net in a curvature metric is in general a non optimal mesh. In constructing upper bounds on the approximation error we have to assume that the vertices of the net are placed as bad as possible. So the first reason is inherent to the method which we use for constructing these bounds. However another reason might be that the constant in Lemma 6.1 is an overestimate.

6.2 Special Cases

In this section we will discuss special cases of Theorem 6.3. These specializations can often be stated in a simpler form than the general theorem. In some of these special cases results are already known, which means that we can make a comparison.

6.2.1 Specialization to Hypersurfaces

In Section 4.6 we discuss the specialization of curvature tensor fields and the related concepts to hypersurfaces. We can use these results to specialize Theorem 6.3 to the case where the embedded manifold is a hypersurface.

Corollary 6.4. *Let M be a compact smooth d-hypersurface with $1 \le d \le 2$. For each $\delta > 0$ and each sufficiently large n, there exists an intrinsic Delaunay mesh $(\mathcal{M}_n^\delta, h)$ of $(M, \mathrm{d}_\nu^\delta)$ with as vertex set a net in $(M, \mathrm{d}_\nu^\delta)$ of cardinality n, such that*

$$\mathrm{d}_H(M, |\mathcal{M}_n^\delta|) \lesssim 8 \left(\frac{\lambda_d \int_M \sqrt{|K(p)|}\, \mathrm{d}\sigma(p)}{nV_d} \right)^{2/d} \quad \textit{as} \quad n \to \infty \quad \textit{and} \quad \delta \downarrow 0.$$

The key step in this specialization is the substitution of $\mu_\nu(M)$ by $\int_M \sqrt{|K(p)|}\, \mathrm{d}\sigma(p)$. Since the Gaussian curvature is, by Gauss's Theorema Egregium, an intrinsic property of the manifold, this upper bound depends only on intrinsic properties of the manifold. This can be explained as follows: In the codimension one case the manifold can only curve in one direction, and the intrinsic properties of the manifold provide enough information to give a good upper bound on the approximation error. In higher codimension the manifold can curve in multiple directions, and the intrinsic information is not enough to provide a good upper bound. In Section 6.3.2 we show why the Gaussian curvature is of no use in an upper bound formula in higher codimension.

Since our result is a generalization to higher codimension of the method in [3], the specialization to hypersurfaces in Corollary 6.3 yields a bound of the same order as in this reference. We have the asymptotically tight result

$$\mathrm{d}_H(M, |\mathcal{M}_n|) \sim \frac{1}{2} \left(\frac{\lambda_d \int_M \sqrt{|K(p)|}\, \mathrm{d}\sigma(p)}{nV_d} \right)^{2/d} \quad \text{as} \quad n \to \infty,$$

for convex hypersurfaces M [15]. Since we also give the constant in the asymptotic notation explicitly, we are able to compare our result to this asymptotically tight result, and we see that if the hypersurface bounds a strictly convex region, then our upper bound is a factor 16 larger than necessary. See the discussion at the end of Section 6.1 for an explanation of this difference.

6.2.2 Specialization to Level Sets

In practice it is often convenient to define a manifold as the level set of a set of functions. In this section we will show that Theorem 6.3 can be applied if a manifold is defined in this way. Hence we show that we are able to compute upper bounds on the meshing approximation error in many practical situations.

A set of functions $f_1, \ldots, f_c : \mathbb{R}^{d+c} \to \mathbb{R}$ is called *linearly independent* on a set $S \subset \mathbb{R}^{d+c}$ if their gradient vectors $\triangledown f_1(p), \ldots, \triangledown f_c(p)$ are linearly independent for all $p \in S$. If M is the zero level set a set of smooth functions $f_1, \ldots, f_c$, that is,

$$M = f_1^{-1}(\{0\}) \cap \ldots \cap f_c^{-1}(\{0\}),$$

such that these functions are linearly independent on M, then M is a smooth d-manifold embedded in $\mathbb{R}^{d+c}$ [10, Theorem 5.22].

Since $\triangledown f_1(p), \ldots, \triangledown f_c(p)$ is a set of smoothly varying linearly independent normal vectors, by Lemma 4.4, the Gram-Schmidt orthonormalization of this set provides a global orthonormal normal frame field ν on M. Hence we have that for a level set manifold M, we can obtain an upper bound on the approximation error by applying Theorem 6.3 with ν as orthonormal normal frame field.

6.2.3 Specialization to Space Curves

The approximation of a curve by a piecewise linear curve, or in other words, the meshing of 1-manifolds, is a well studied topic. In the following lemma we state a result by Fejes Tóth, which shows that the asymptotic approximation error of an optimal mesh of a space curve can be expressed in the curvature of the curve. Given a smooth space curve $\gamma : [a, b] \to \mathbb{R}^{1+c}$, the *curvature* $\kappa_\gamma(t)$ at $\gamma(t)$ is defined by

$$\kappa_\gamma(t) = \frac{\|\gamma'(t) \times \gamma''(t)\|}{\|\gamma'(t)\|^3}.$$

This definition is invariant under reparametrizations of γ and is chosen such that the radius of the circle of curvature at $\gamma(t)$ equals $1/\kappa_\gamma(t)$.

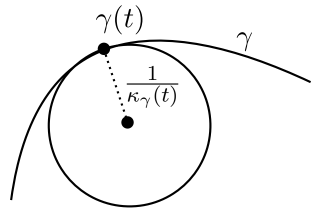

Figure 6.1: The curvature $\kappa_\gamma(t)$ at t of a curve γ.

Theorem 6.5 ([16]). *Let M be a smooth compact connected 1-manifold embedded in $\mathbb{R}^{1+c}$ and let*

$$\gamma : [0, l] \to \mathbb{R}^{1+c}$$

be a parametrization by arc length of M. If $\mathcal{M}_n$ are optimal, with respect to the Hausdorff distance, meshes of M, where n is the cardinality of the vertex set, then

$$\mathrm{d}_H(M, |\mathcal{M}_n|) \sim \frac{1}{8n^2} \left(\int_0^l \sqrt{|\kappa_\gamma(s)|}\, \mathrm{d}s \right)^2 \quad as \quad n \to \infty.$$

We will use the final c vectors of the Frenet-Serret frame of the curve as orthonormal normal frame field ν, and then specialize Theorem 6.3 to the $d = 1$ case. Then we will compare the result to the above asymptotically tight result.

Lemma 6.6. *Let M be a smooth compact connected 1-manifold embedded in $\mathbb{R}^{1+c}$ and let*

$$\gamma : [0, l] \to \mathbb{R}^{1+c}$$

be a parametrization by arc length of M. If $\mathcal{M}_n$ are optimal, with respect to the Hausdorff distance, meshes of M, where n is the cardinality of the vertex set, then

$$\mathrm{d}_H(M, |\mathcal{M}_n|) \lesssim \frac{2}{n^2} \left(\int_0^l \sqrt{|\kappa_\gamma(s)|}\, \mathrm{d}s \right)^2 \quad as \quad n \to \infty.$$

Proof. Let $\eta^1_{\gamma(t)}, \nu^1_{\gamma(t)}, \ldots, \nu^c_{\gamma(t)}$ be the Gram-Schmith orthonormalization of the vectors

$$\gamma'(t), \gamma''(t), \ldots, \gamma^{(1+c)}(t).$$

The frame $\eta_{\gamma(t)} = \{\eta^1_{\gamma(t)}\}$ is an orthonormal tangent frame at $\gamma(t) \in M$, and the frame $\nu_{\gamma(t)} = \{\nu^1_{\gamma(t)}, \ldots, \nu^c_{\gamma(t)}\}$ is an orthonormal normal frame at $\gamma(t) \in M$. The adapted frame $(\eta_{\gamma(t)}, \nu_{\gamma(t)})$ is called the Frenet-Serret frame of M at $\gamma(t)$. By Lemma 4.4, η and ν are frame fields of M.

Since the second order behaviour of the curve is in the direction of the osculating plane,

$$C(\eta_{\gamma(t)}; \nu^i_{\gamma(t)}) = 0 \quad \text{for all} \quad 2 \le i \le c.$$

If a curve is locally the graph of a function $g : \mathbb{R} \to \mathbb{R}$, then the curvature at $g(t)$ is given by

$$\frac{g''(t)}{(1 + (g'(t))^2)^{3/2}}.$$

Since $(f(\eta_{\gamma(t)}; \nu^1_{\gamma(t)}))'(0) = 0$ we have that

$$C(\eta_{\gamma(t)}; \nu^1_{\gamma(t)}) = \left| (f(\eta_{\gamma(t)}; \nu^1_{\gamma(t)}))''(0) \right| = |\kappa_\gamma(t)|.$$

Combining the above result we have that

$$C(\eta_{\gamma(t)}; \nu_{\gamma(t)} t) = C(\eta_{\gamma(t)}; \nu^1_{\gamma(t)}) = |\kappa_\gamma(t)|,$$

and therefore

$$\mu_\nu(M) = \int_M \sqrt{\det\left(C(\eta_{\gamma(t)}; \nu_{\gamma(t)})\right)} \mathrm{d}s = \int_M \sqrt{|\kappa_\gamma(s)|} \mathrm{d}s.$$

The volume of the 1-dimensional unit ball $[-1, 1] \subset \mathbb{R}$ is $V_1 = 2$, and the packing density of $\mathbb{R}$ is $\lambda_1 = 1$. We can complete the proof by applying Theorem 6.3, where we use ν as orthonormal normal frame field. □

This means that in just as in the case of convex hypersurfaces, the upper bound which we have obtained in this report is a factor 16 larger than the optimal value. See the discussion at the end of Section 6.1 for an explanation of this difference.

6.3 Examples

In this section we will use Theorem 6.3 or its specialization to compute upper bounds on the approximation error for specific embedded manifolds. In these examples we start by finding an orthonormal normal frame field ν and we will use this to compute the curvature numbers of the manifold. Using these curvature numbers we obtain the value of $\mu_\nu(M)$ and use this in the upper bound formula.

6.3.1 Hyperspheres

The *d-hypersphere* with radius R, centered about the origin, is given by

$$\mathbb{S}_R^d = \{x \in \mathbb{R}^{d+1} \mid x_1^2 + \ldots + x_{d+1}^2 = R^2\}.$$

The orthonormal normal frame field yielded by the inward normal section is given by $\nu_p = \{\nu_p^1\}$ with $\nu_p^1 = -p/R$. Due to the symmetry of the sphere we have that $e(\eta_p; \nu_p)\mathbb{S}_R^d$ is the same for each $p \in \mathbb{S}_R^d$ and each orthonormal tangent frame η_p. The manifold $e(\eta_p; \nu_p)\mathbb{S}_R^d$ is locally parametrized by

$$V_0 \to \mathbb{R}^{d+c}, \quad (x_1, \ldots, x_d) \mapsto \left(x_1, \ldots, x_d, R - \sqrt{R^2 - x_1^2 - \ldots - x_d^2}\right),$$

where V_0 is an open neighbourhood of $0 \in \mathbb{R}^d$. The function $f(\eta_p, \nu_p^1)$ is given by

$$f(\eta_p, \nu_p^1)(x) = R - \sqrt{R^2 - x_1^2 - \ldots - x_d^2},$$

so

$$Hf(\eta_p, \nu_p^1)(0) = \frac{1}{R}I.$$

Since these eigenvalues are all non-negative we do not have to convexify and

$$C(\eta_p, \nu_p) = C_{\eta_p}^{\nu_p^1} = Hf(\eta_p, \nu_p^1)(0) = \frac{1}{R}I.$$

Using that the surface area of $\mathbb{S}_R^d$ is $2\pi R^d V_{d-1}$, we have

$$\mu_\nu(\mathbb{S}_R^d) = \int_{\mathbb{S}_R^d} \sqrt{|\det(C(\eta_p, \nu_p))|}\mathrm{d}A = \frac{1}{R^{d/2}} \int_{\mathbb{S}_R^d} dA = 2\pi R^{d/2} V_{d-1}.$$

So, by Theorem 6.3, for n sufficiently large and $\delta > 0$ sufficiently small there exists simplicial complexes $(\mathcal{S}_R^d)_n^\delta$ with $V((\mathcal{S}_R^d)_n^\delta) \subset \mathbb{S}_R^d$ an ϵ-net in $(\mathbb{S}_R^d, \mathrm{d}_\nu^\delta)$, such that

$$\mathrm{d}_H(\mathbb{S}_R^d, (\mathcal{S}_R^d)_n^\delta) \lesssim 8R \left(\frac{2\pi\lambda_d V_{d-1}}{V_d n}\right)^{2/d} \quad \text{as} \quad n \to \infty \quad \text{and} \quad \delta \downarrow 0.$$

Low dimensional examples. Since $V_0 = 1$, $V_1 = 2$, and $\lambda_1 = 1$ we have that

$$\mathrm{d}_H(\mathbb{S}_R^1, (\mathcal{S}_R^1)_n^\delta) \lesssim 8\pi^2 R \frac{1}{n^2} \quad \text{as} \quad n \to \infty \quad \text{and} \quad \delta \downarrow 0.$$

By integrating the root of the constant curvature $1/R$ over the circle we see that Lemma 6.6 yields the same result as above. Since $V_1 = 2$, $V_2 = \pi$, and $\lambda_2 = \frac{\pi}{2\sqrt{3}}$ we have that

$$\mathrm{d}_H(\mathbb{S}_R^2, (\mathcal{S}_R^2)_n^\delta) \lesssim \frac{16\pi}{\sqrt{3}} \frac{R}{n} \quad \text{as} \quad n \to \infty \quad \text{and} \quad \delta \downarrow 0.$$

6.3.2 Flat Tori

A global parametrization of the *Clifford torus* T with radii $R_1 > 0$ and $R_2 > 0$, embedded in $\mathbb{R}^4$, is given by

$$\mathcal{X}(\theta, \phi) = (R_1 \cos(\theta), R_1 \sin(\theta), R_2 \cos(\phi), R_2 \sin(\phi)),$$

where θ and ϕ range from 0 to 2π. Note that a Clifford torus with radii R_1 and R_2 is contained in $\mathbb{S}^3_R$, where $R = \sqrt{R_1^2 + R_2^2}$.

Since a Clifford torus is a *flat manifold*, that is, its Gaussian curvature is zero everywhere, it is also called a *flat torus*. This illustrates that we cannot use the Gaussian curvature for an upper bound in higher codimension.

An orthonormal normal frame field of T is given by $\nu = \{\nu^1, \nu^2\}$, with

$$\nu^1_{\mathcal{X}(\theta,\phi)} = (-\cos(\theta), -\sin(\theta), 0, 0) \quad \text{and} \quad \nu^2_{\mathcal{X}(\theta,\phi)} = (0, 0, -\cos(\phi), -\sin(\phi)).$$

An orthonormal tangent frame field of T is given by $\eta = \{\eta^1, \eta^2\}$, with

$$\eta^1_{\mathcal{X}(\theta,\phi)} = \frac{\partial}{\partial\theta}\mathcal{X}(\theta, \phi) = (-\sin(\theta), \cos(\theta), 0, 0)$$

and

$$\eta^2_{\mathcal{X}(\theta,\phi)} = \frac{\partial}{\partial\phi}\mathcal{X}(\theta, \phi) = (0, 0, -\sin(\phi), \cos(\phi)).$$

Recall from Section 4.2 that $e(\eta_p, \nu_p)$ is the Euclidean transformation on $\mathbb{R}^4$ such that $e(\eta_p, \nu_p)p = 0$ and

$$(e(\eta_p, \nu_p)\eta^1_p, e(\eta_p, \nu_p)\eta^2_p, e(\eta_p, \nu_p)\nu^1_p, e(\eta_p, \nu_p)\nu^2_p) = (e_1, e_2, e_3, e_4).$$

Due to the symmetry of the Clifford torus, $e(\eta_p, \nu_p)T$ is the same for each $p \in T$, and is locally parametrized by

$$(s, t) \mapsto \left(s, t, R_1 - \sqrt{R_1^2 - s^2}, R_2 - \sqrt{R_2^2 - t^2}\right),$$

in an open neighbourhood V_0 of $0 \in \mathbb{R}^2$. So,

$$f^{\nu^1_p}_{\eta_p}(s, t) = R_1 - \sqrt{R_1^2 - s^2} \quad \text{and} \quad f^{\nu^2_p}_{\eta_p}(s, t) = R_2 - \sqrt{R_2^2 - t^2},$$

hence

$$C(\eta_p, \nu_p) = C(\eta_p, \nu^1_p) + C(\eta_p, \nu^2_p) = \left|Hf^{\nu^1_p}_{\eta_p}(0, 0)\right| + \left|Hf^{\nu^2_p}_{\eta_p}(0, 0)\right| = \begin{pmatrix} 1/R_1 & 0 \\ 0 & 1/R_2 \end{pmatrix}.$$

Using that the surface area of T is $4\pi^2 R_1 R_2$, we have

$$\mu_\nu(T) = \int_T \sqrt{\det(C(\eta_p, \nu_p))}\mathrm{d}A = \frac{1}{\sqrt{R_1 R_2}} \int_T dA = 4\pi^2\sqrt{R_1 R_2}.$$

So, by Theorem 6.3, for n sufficiently large and $\delta > 0$ sufficiently small there exists simplicial complexes $\mathcal{T}^\delta_n$ with $V(\mathcal{T}^\delta_n) \subset T$ an ϵ-net in $(T, \mathrm{d}^\delta_\nu)$, such that

$$\mathrm{d}_H(T, \mathcal{T}^\delta_n) \lesssim \frac{32\pi^2}{\sqrt{3}} \frac{\sqrt{R_1 R_2}}{n} \quad \text{as} \quad n \to \infty \quad \text{and} \quad \delta \downarrow 0. \tag{6.1}$$

Minimality. For a given p, the number $\det(C(\eta_p, \nu_p))$ depends on the normal frame ν_p at p, which raises the question whether we could have obtained a better bound by choosing a different normal frame field. We will show that for each p, the normal frame ν_p as defined above minimizes $\det(C(\eta_p, \nu_p))$ over all orthonormal normal frames at p, and hence that ν is an optimal normal frame field.

For some fixed p, let

$$r_\alpha = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos(\alpha) & -\sin(\alpha) \\ 0 & 0 & \sin(\alpha) & \cos(\alpha) \end{pmatrix}.$$

By repeating the calculation above with $r_\alpha \nu$ instead of ν, we obtain

$$\det(C(\eta_p, r_\alpha \nu_p)) = \frac{1 + |\sin(2\alpha)|}{R_1 R_2},$$

which is independent of the point p, and is minimal for each $\alpha = k\pi/2$ with $k \in \mathbb{Z}$. In particular it is minimal for $\alpha = 0$, so the normal frame field ν is indeed optimal.

Chapter 7

Curvature Tensor Fields and Second Fundamental Forms

Curvature tensor fields are defined using the Hessian matrices of a set of height functions of which the manifold is locally the graph. In codimension one there is a canonical curvature tensor field which is also called the second fundamental form. In this chapter we will give general definitions of the second fundamental forms and shape operators, and we will specialize these definitions to the case of Euclidean ambient spaces and the case of hypersurfaces. We will then give the relation between curvature tensor fields, second fundamental forms, and shape operators.

7.1 Second Fundamental Forms and Shape Operators

We will give the definitions of the second fundamental forms and shape operators in the general setting of Riemannian geometry. We will then specialize these definitions to the case of manifolds embedded in Euclidean spaces and to the case of hypersurfaces. In the literature there are many different but related ways to define the second fundamental forms; we will use some definitions from [18].

Let M be a differentiable d-manifold embedded in a Riemannian $(d+c)$-manifold $\overline{M}$. The tangent and normal bundles are related by

$$T\overline{M}|_M = TM \oplus NM.$$

Let $\top$ and $\perp$ be the projections on the tangent and the normal bundle and denote by $\Gamma(F)$ the space of smooth sections on a fibre bundle F.

Denote the inner product on the tangent spaces of $\overline{M}$ by $\langle .,. \rangle$ and note that the Riemannian structure of $\overline{M}$ induces a Riemannian structure on the submanifold M. Let $\overline{\nabla}$ be the Levi-Civita connection on $\overline{M}$. In [18] it is shown that the Levi-Civita connection on M is given by

$$\nabla_u v = (\overline{\nabla}_u v)^\top,$$

for $u, v \in \Gamma(TM)$.

We define the *second fundamental map* of M in $\overline{M}$ as

$$B : TM \times TM \to NM, \quad (u, v) \mapsto (\overline{\nabla}_u v)^\perp = \overline{\nabla}_u v - \nabla_u v.$$

In the following lemma we see that this is a bilinear map, and that this map is symmetric, as the notation suggests.

Lemma 7.1. *The normal second fundamental map B of M in $\overline{M}$ is a symmetric bilinear map.*

Proof. Let $u, v, w \in \Gamma(TM)$ and $f, g \in \mathcal{F}(M)$. The Levi-Civita connection is torsion free, that is, $\nabla_u v - \nabla_v u = [u, v]$ and $\overline{\nabla}_u v - \overline{\nabla}_v u = [u, v]$. So

$$B(u, v) = \overline{\nabla}_u v - \nabla_u v = \nabla_v u + [u, v] - \overline{\nabla}_v u - [u, v] = B(v, u).$$

Since the Levi-Civita connection is a covariant derivative we also have

$$\begin{aligned} B(fu + gv, w) &= (\overline{\nabla}_{fu+gv} w)^{\perp} \\ &= (f\overline{\nabla}_u w + g\overline{\nabla}_v w)^{\perp} \\ &= f(\overline{\nabla}_u w)^{\perp} + g(\overline{\nabla}_v w)^{\perp} \\ &= fB(u, w) + gB(v, w). \end{aligned}$$

□

For $\nu \in \Gamma(NM)$, we use the second fundamental map to define the *second fundamental form* s_ν by

$$s_\nu : TM \times TM \to \mathbb{R}, (u, v) \mapsto -\langle B(u, v), \nu \rangle.$$

It follows from Lemma 7.1 that a second fundamental form is a symmetric bilinear form.

For $\nu \in \Gamma(NM)$, the *shape operator* is defined as

$$A^\nu : TM \to TM, \quad u \mapsto -(\overline{\nabla}_u \nu)^{\top}.$$

The following lemma shows the relation between a shape operator and a second fundamental map/form.

Lemma 7.2 (Weingarten Equations)**.** *For each $u, v \in \Gamma(TM)$ and $\nu \in \Gamma(NM)$,*

$$s_\nu(u, v) = -\langle B(u, v), \nu \rangle = \langle A^\nu(u), v \rangle.$$

Specialization to Euclidean ambient spaces. When the ambient space $\overline{M}$ is a Euclidean space $\mathbb{R}^{d+c}$, then we can replace the covariant derivatives in the definitions by normal derivatives. We have that

$$(\overline{\nabla}_u v)_p = Dv(p)u_p$$

for $u, v \in \Gamma(T\overline{M})$. In the notation $Dv(p)$, we view v as a column vector consisting of functions which depend on p, therefore $Dv(p)$ is a matrix depending on p. The right hand side of this equation confirms that the covariant derivative $(\overline{\nabla}_u v)_p$ depends on the vector field u only at the point p, but on the vector field v in an open neighbourhood of p. The ith component of the vector $Dv(p)u_p$ shows how much the ith component of $v(p)$ changes when moving in the direction u_p.

We have that

$$B(u, v)_p = (\overline{\nabla}_u v)_p^{\perp} = (Dv(p)u_p)^{\perp}$$

and

$$(A^\nu(u))_p = -(\overline{\nabla}_u \nu)_p^{\top} = -(D\nu(p)u_p)^{\top}.$$

The ith component of the vector $B(u, v)_p$ shows how much the ith component of $v(p)$ changes along the normal plane N_pM when moving in the direction u_p. The ith component of the vector $(A^\nu(u))_p$ shows how much the ith component of the normal $\nu(p)$ changes along the tangent plane T_pM when moving in the direction $-u_p$.

7.2 Convex Hypersurfaces

A d-hypersurface of class C^k is called a *convex hypersurface* if it is the boundary of some compact strictly convex $(d+1)$-manifold embedded in $\mathbb{R}^{d+1}$ with boundary of class C^k. In Section 4.6 we have seen that if M is a hypersurface with a unit normal field ν, then $\det((g^0_\nu)_p) = K(p)$ for all $p \in M$. It follows from the fact that a convex hypersurface has positive Gaussian curvature everywhere that g^0_ν is a Riemannian tensor field of class C^k. In the following lemma we will show that for a convex hypersurface this Riemannian tensor field is equal to the second fundamental form s_ν.

Lemma 7.3. *If M is a convex hypersurface of class C^2, and if $\nu \in \Gamma(NM)$ is a global unit normal section, then the fundamental form s_ν is equal to the curvature tensor field q^0_ν.*

Proof. Let $p \in M$ and $u, v \in \Gamma(TM)$. We may assume without loss of generality that $p = 0$ and T_0M is the $x_{d+1} = 0$ plane. Let $\eta_0 = \{e_1, \ldots, e_d\}$. By Lemma 4.5 there is a neighbourhood U_0 of 0 such that U_0 is given by the implicit equation

$$F(x_1, \ldots, x_{d+1}) = x_{d+1} - f(x_1, \ldots, x_d) = 0,$$

for some $f : \mathbb{R}^d \to \mathbb{R}$, such that the inward unit normal section is given by

$$\begin{aligned} \nu(x_1, \ldots, x_{d+1}) &= \text{normalize}(\nabla F(x_1, \ldots, x_{d+1})) \\ &= \text{normalize}((-Df(x_1, \ldots, x_d), 1)^T). \end{aligned}$$

We then have

$$\begin{aligned} (s_\nu)_0(u_0, v_0) &= -\langle B(u, v), \nu\rangle_0 \\ &= -\langle (\bar{\nabla}_u v)^\perp, \nu\rangle_0 \\ &= -\langle \bar{\nabla}_u v, \nu\rangle_0 \\ &= -\langle v, \bar{\nabla}_u \nu\rangle_0 \\ &= -u_0^T D\nu(0) v_0 \\ &= -u_0^T v_0 \\ &= [u_0]^T_{\eta_0} Hf(0)[v_0]_{\eta_0} \\ &= [u_0]^T_{\eta_0} C(\eta_0, \nu_0)[v_0]_{\eta_0} \\ &= (q^\nu_0)_0(u_0, v_0), \end{aligned}$$

which implies that $s_\nu = q^0_\nu$. □

7.3 Manifolds Embedded in $\mathbb{R}^n$

When a hypersurface is not a convex hypersurface then it is not possible to choose an orientation ν of the hypersurface such that the second fundamental form s_ν is positive definite everywhere. Since a curvature tensor field q^0_ν is positive semidefinite everywhere, the result of Lemma 7.3 cannot hold for non convex hypersurfaces. The reason for this is that we need to convexify the second fundamental form. We will show how this can be done in the general case of a smooth d-manifold M embedded in $\mathbb{R}^{d+c}$ which is equipped with a global orthonormal normal frame field $\nu = \{\nu^1, \ldots, \nu^c\}$.

For each $p \in M$, the second fundamental form $(s_{\nu^i})_p$ is a bilinear map from T_pM to $\mathbb{R}$. By choosing a basis η_p of T_pM we obtain a matrix A_p such that that $(s_{\nu^i})_p(u_p, v_p) = [u_p]_{\eta_p}^T A_p [v_p]_{\eta_p}$ for each $u_p, v_p \in T_pM$. We now define a new map

$$(g_\nu)_p(u, v) = [u_p]_{\eta_p}^T |A_p| [v_p]_{\eta_p},$$

where $|A_p|$ is the absolute value of the matrix as discussed in Section 2.2. It can be shown that this map is independent of the choice of basis η_p, which is similar to Lemma 4.8. In the following lemma we express the curvature tensor field q_ν^0 in a sum of these bilinear maps g_{ν^i}. We will omit the proof of this lemma since it is similar to the proof of Lemma 7.3.

Lemma 7.4. *If M is a smooth d-manifold embedded in $\mathbb{R}^{d+c}$ and $\nu = \{\nu^1, \ldots, \nu^c\}$ is a global orthonormal normal frame field, then*

$$q_\nu^0 = \sum_{i=1}^{c} g_{\nu^i}.$$

Chapter 8

Future Work

In this chapter we discuss some open problems which might be interesting for future research. First we discuss the generalization to manifolds with boundary, which might be useful in removing the condition that a manifold should admit a nonoriented orthonormal normal frame. There are some results in this report which we have only stated and proven in low dimensions, we discuss how this prevents us from stating the main theorem of the report in full generality, and how this might be resolved. In this report we have focused on the construction of upper bounds, in Section 8.3 we will briefly discuss the more difficult problem of constructing lower bounds. Finally, in Section 8.4 we will briefly discuss some computational considerations.

8.1 Manifolds with Boundary

In this report we have only considered manifolds without boundary. Since the boundary of manifolds of codimension 1 or higher is at least 2, the generalization to manifolds of higher codimension is a step in the generalization to manifolds with boundary.

In special cases the boundary of a manifold of higher codimension can be considered to be a hypersurface. As an illustration we will discuss two of these instances. Consider a unit disk D and a unit hemisphere H embedded in $\mathbb{R}^3$. The boundary of each is a unit circle $\mathbb{S}^1$, and by the example of Section 6.3.1 there exist meshes $\mathcal{S}_n^\delta$ of this circle such that

$$\mathrm{d}_H(\mathbb{S}^1, \mathcal{S}_n^\delta) \lesssim 50\pi^2 \frac{1}{n^2} \quad \text{as} \quad n \to \infty \quad \text{and} \quad \delta \downarrow 0.$$

By taking a single vertex in the centre of the disk we can extend $\mathcal{S}_n^\delta$ to a mesh $\mathcal{D}_n^\delta$ of D. If n grows, then each interior point of D will eventually be covered by a mesh simplex, so

$$\mathrm{d}_H(\mathrm{int}(D), \mathcal{D}_n^\delta) = 0 \quad \text{as} \quad n \to \infty \quad \text{and} \quad \delta \downarrow 0.$$

So we obtain the upper bound

$$\mathrm{d}_H(D, \mathcal{D}_n^\delta) \lesssim 50\pi^2 \frac{1}{n^2} \quad \text{as} \quad n \to \infty \quad \text{and} \quad \delta \downarrow 0.$$

The case of the hemisphere is more complex and needs generalizations of the theorems to manifolds with boundary. Since the boundary of a manifold with boundary is a manifold, we can use Theorem 6.3 for the boundary. In the ideal case it would be possible to show that for all $\delta > 0$ and all sufficiently large n there exists a mesh $\mathcal{M}_n^\delta$ of a manifold with boundary M as given in Theorem 6.3 such that the simplicial boundary of this mesh is a mesh of the boundary as given in Theorem 6.3.

Delaunay triangulations of manifolds with boundary. In order to generalize Theorem 6.3 to manifolds with boundary we will need to generalize the lemmas concerning Delaunay triangulations of manifolds; in particular we need to generalize Lemma 5.2. We will start by showing why the proof sketch that Leibon and Letcher give in [11, Section 5] does not generalize easily to manifolds with boundary.

Let M be a Riemannian manifold with induced metric d. The proof works by first showing that the Voronoi diagram for a sufficiently dense set of points $\{x_1, \ldots, x_n\} \subset M$ is a cell decomposition of M. Then it is shown that the dual of this diagram is a Delaunay triangulation of M with vertex set $\{x_1, \ldots, x_n\}$. The dual of this Voronoi diagram will be the Delaunay triangulation. The map

$$f : M \to \mathbb{R}^n,\ p \mapsto (\mathrm{d}(p, x_1), \ldots, \mathrm{d}(p, x_n))$$

is used to study this Voronoi diagram. Note that the Voronoi region of x_i is given by

$$\{p \in M \mid d(p, x_i) \leq d(p, x_j) \text{ for all } 1 \leq j \leq n\} = f^{-1}(C_i),$$

where

$$C_i = \{(z_1, \ldots, z_n) \mid 0 \leq z_i \leq z_j \text{ for all } 1 \leq j \leq n\}.$$

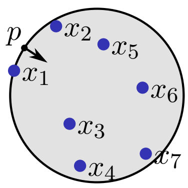

Figure 8.1: Moving from p towards the centre of the disk decreases the distance to each x_i

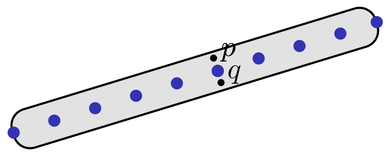

Figure 8.2: The map f is not injective since $f(p) = f(q)$

It is shown that this map f is an embedding of the manifold in $\mathbb{R}^n$. To show this, the proof uses the fact that moving in any direction from any point will always increase the distance to one of the points in $\{x_1, \ldots, x_n\}$. In Figure 8.1 we see that this is not true for manifolds with boundary, even for arbitrarily dense point sets. This means that we cannot use this argument to show that f is injective. In fact, in Figure 8.2 we show that there exist manifolds with boundary for which there exist point sets that satisfy the density radius condition but where the map f is not injective. However, we expect that for a sufficiently dense point set with some mild additional conditions, the map f can be shown to be injective, and hence can be shown to be an embedding. So it might be possible to modify the proof to work for manifolds with boundary.

Cutting a manifold in small pieces. In Theorem 6.3 we require the manifold to be equipped with a nonoriented orthonormal normal frame field ν. The curvature numbers $c_\nu(p)$ depend on this frame field and the value $\mu_\nu(M) = \int_M c_\nu(p)\,\mathrm{d}\sigma(p)$ is used in the upper bound on the approximation error. The extension of the results of this report to manifolds with boundary might be useful to remove the requirement that the manifold admits a nonoriented orthonormal frame field ν. This might be possible by cutting the manifold in many small manifolds with boundary, since for these small patches orthonormal normal frames do exists. A problematic part is the patching together of the resulting meshes to obtain one large mesh of the original manifold. If it could be shown that the above is possible, then it might also be possible to replace $\mu_\nu(M)$ in the upper bound formula of Theorem 6.3 by $\int_M c(p)\,\mathrm{d}\sigma(p)$, where $c(p)$ is a minimal curvature number at p as defined at the end of Section 4.3. Since $c(p)$ can be computed point wise, this might aid in the computation of these upper bounds.

8.2 Higher Dimensions

There are some results in Chapter 5 for which we require the dimension d to be 1 or 2, and there is a result where we additionally require the codimension c to be 1. The first result which we have not proven in full generality is about diffeomorphic images of small disks; the proof of Lemma 5.6 works only in the planar case. In particular, we use that the existence of an injective unit tangent vector field implies the existence of an injective unit normal vector field. However, we expect that this proof can be modified such that it works for arbitrary dimensions $d \geq 1$.

The second result where we require the dimension d of the manifold to be 1 or 2 is Lemma 5.8. The proof of this lemma does not generalize easily to higher dimensions. It uses for instance that in a Delaunay triangulation the vertices of a triangle do not all lie near a line. However, in the case where $d = 3$ there can be so called *slivers* in a Delaunay triangulation, where the four points of a simplex lie close to a hyperplane. Currently the lemma states that for any Delaunay triangulation with a sufficiently dense net as vertex set, the local subcollections are simplicial complexes. For dimensions higher than 2 it might be necessary to change this to slightly weaker statement: for any sufficiently small $\epsilon > 0$, there exists an ϵ-net and a Delaunay triangulation with this net as vertex set, such that the local subcollections are simplicial complexes.

The third result which we have only stated in low dimensions and in fact also only in codimension 1 is Corollary 5.15. The reason for this is that this result depends on Lemma 5.14 for which in the reference is only stated for $d = 2$ and $c = 1$. However, we expect no problems in generalizing Lemma 5.14 to higher dimension and higher codimension.

8.3 Lower Bounds

In this report we have focused on upper bounds, but it is expected that the main ideas of this method can be used in the more difficult problem of constructing lower bounds. In [3, Section 4.2] Clarkson discusses the construction of such lower bounds, but a fully general solution has not yet been given. A problem with lower bounds is that for nonconvex manifolds the lower bounding version of Lemma 4.14 does not hold. Another problem is that by using intrinsic Delaunay meshes with as vertex sets nets in curvature metrics, all simplices will be small. An optimal mesh, however, might require the occurrence of large simplices. Note that the lower bounding part of Corollary 3.10, which uses the minimal sphere covering density numbers, and Lemma 4.12, which we did not need in the construction of upper bounds, can be of use in the construction of lower bounds.

8.4 Computational Aspects

The construction of meshes using curvature metrics, nets, and Delaunay triangulations might be interesting from a computational perspective. Future research could be done into the computational feasibility of this method for mesh construction. However, since it is generally hard to do computations in intrinsic metrics, it is questionable whether this yields sufficiently fast algorithms. In [1] Boissonnat and Ghosh give a meshing algorithm for higher codimension which does not require these intrinsic calculations. An upper bound on the approximation error is also given, which depends on the reach of the manifold, but where the constant in the asymptotic bounds is not given explicitly.

We have used these curvature metrics, nets, and Delaunay triangulations to *prove* results about upper bounds on the approximation errors. However, to *compute* these upper bounds, we only need curvature measures. From a computational perspective, curvature measures are easier to deal with than curvature metrics. Future research could be done on the computation of $\mu_\nu(M)$, where μ_ν is a curvature measure and ν a nonoriented orthonormal normal frame field of M. In particular, we show in Section 6.2.2 that any manifold M which is the level set of smooth functions admits an orthonormal normal frame field ν. This gives a method to compute $\mu_\nu(M)$ for a manifold which is a level set, and hence we can compute an upper bound on the approximation error. So it might be of particular interest to investigate the computational feasibility of finding asymptotic upper bounds on the approximation error of level sets.

List of Symbols

$\lesssim$	Asymptotic inequality, page 5
d_ν^δ	Curvature metric with respect to a nonoriented orthonormal normal frame field ν and a positive scalar δ, page 24
μ_ν^δ	Curvature measure with respect to an orthonormal normal frame field ν and a positive scalar δ, page 25
π_M	Projection along the fibres of a tubular neighbourhood, page 42
τ_{η_p}	A chart of the neighbourhood U_p of p with as basis the orthonormal tangent frame η_p, page 19
lsc_p	Local subcollection, page 37
$C(\eta_p, \nu_p)$	Curvature matrix with respect to an orthonormal tangent frame η_p and an orthonormal normal frame ν_p, page 21
$c_\nu^\delta(p)$	Curvature number at the point p with respect to an orthonormal normal frame field ν and a positive scalar δ, page 22
$e(\eta_p, \nu_p)$	Euclidean transformation mapping the vectors of the adapted frame (η_p, ν_p) to the standard basis vectors, page 19
E_h	Linear approximation, page 36
$f(\eta_p, \nu_p^i)$	Height function with respect to the orthonormal tangent frame η_p and the normal vector ν_p^i, page 19
q_ν^δ	Curvature tensor field with respect to a nonoriented orthonormal normal frame field ν and a positive scalar δ, page 22
T_M	Tubular neighbourhood around M with radius $\mathrm{reach}(M)$, page 42
U_p	Maximal open neighbourhood of p which can be written as a graph, page 20
hull	Convex hull, page 33

Index

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