A semidefinite programming hierarchy for packing problems in discrete geometry

David de Laat (TU Delft)
Joint work with Frank Vallentin (Universität zu Köln)

Applications of Real Algebraic Geometry
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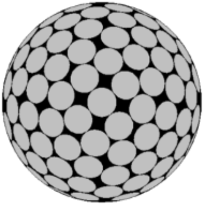
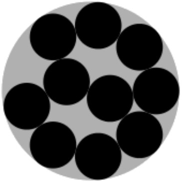
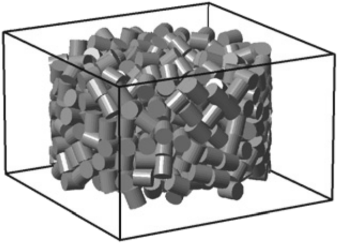
Packing problems in discrete geometry

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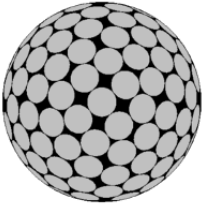
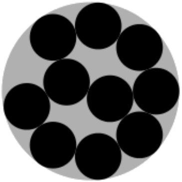
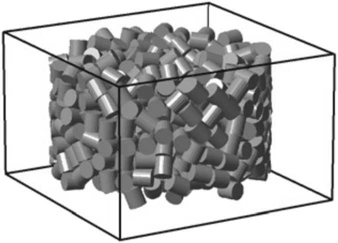

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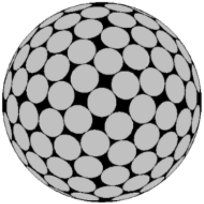
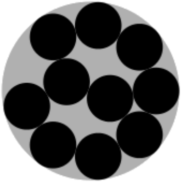
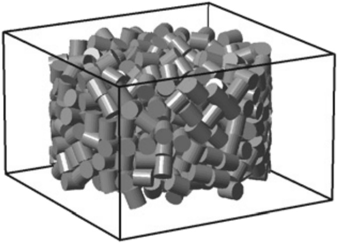

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Spherical cap packings

What is the maximum number of spherical caps of size t in S^{n-1} such that no two caps intersect in their interiors?
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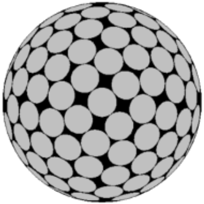

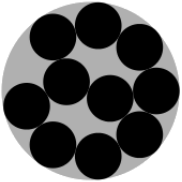

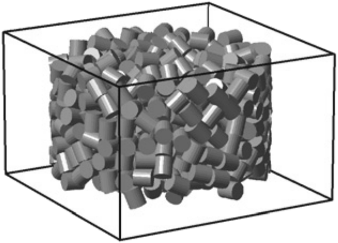

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- Independent sets correspond to valid packings

The Lasserre hierarchy for finite graphs

- Maximum independent set problem for a finite graph as a 0/1 polynomial optimization problem:

$$\alpha(G) = \max\Big\{ \sum_{v \in V} x_v : x_v \in \{0,1\} \text{ for } v \in V,\ x_u + x_v \leq 1 \text{ for } \{u,v\} \in E \Big\}$$

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- $\vartheta'(G) = \mathrm{las}_1(G) \geq \mathrm{las}_2(G) \geq \ldots \geq \mathrm{las}_{\alpha(G)}(G) = \alpha(G)$
- $\vartheta(G)$ is the Lovász ϑ-number which specializes to the Delsarte LP-bound when G is the binary code graph

Generalization to infinite graphs

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- These graphs have finite independence number

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$$(K(J_i, J_j))_{i,j=1}^m \succeq 0 \quad \text{for all} \quad m \in \mathbb{N} \quad \text{and} \quad J_1, \ldots, J_m \in V_t$$

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- Cone of positive definite kernels: $\mathcal{C}(V_t \times V_t)_{\succeq 0}$

- Cone of positive definite measures:

$$\mathcal{M}(V_t \times V_t)_{\succeq 0} = \{\mu \in \mathcal{M}(V_t \times V_t)_{\mathrm{sym}} : \mu(K) \geq 0 \text{ for all } K \in \mathcal{C}(V_t \times V_t)_{\succeq 0}\},$$

where $\mu(K) = \int K(J, J') \, d\mu(J, J')$

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- The adjoint: $A_t^* \colon \mathcal{M}(I_{2t}) \to \mathcal{M}(V_t \times V_t)_{\mathsf{sym}}$

Finite convergence

Theorem

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- Then, $\lambda(I_{=1}) = \int \chi_S(I_{=1}) \, d\sigma(S) = \int |S| \, d\sigma(S) \leq \alpha(G)$

Duality theory

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- These are infinite dimensional conic programs

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- Representation: $O(3) \to \mathcal{L}(\mathcal{C}(V_2))$, $gf(x) = f(g^{-1}x)$

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- Modeling these constraints using sums of squares characterizations reduces the problems to finite dimensional semidefinite programs

Thank you

D. de Laat, F. Vallentin, A semidefinite programming hierarchy for packing problems in discrete geometry, arXiv:1311.3789 (2013), 21 pages.

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