Entanglement dimension and noncommutative polynomial optimization

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Deterministic correlations:

- $P \in [0,1]^\Gamma$ of the form $P(a,b|s,t) = P_\mathcal{A}(a|s)P_\mathcal{B}(b|t)$ with

$$P_\mathcal{A} \in \{0,1\}^{A\times S},\ P_\mathcal{B} \in \{0,1\}^{B\times T},\ \sum_a P_\mathcal{A}(a|s) = \sum_b P_\mathcal{B}(b|t) = 1$$

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Set of classical correlations:

- $C_c(\Gamma)$ = convex hull of deterministic correlations
- Here we assume access to shared randomness

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- $D_q(P) = \min\left\{d^2 : d \in \mathbb{N},\, P \in C_q^d(\Gamma)\right\}$

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- (But if $D_q(P) > |\Gamma| + 1 - |S||T|$, then P is not classical)
- Computing $D_q(P)$ is NP-hard (Stark 2015)

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$$\min\Big\{ \max_{i=1,\dots,I} D_q(P_i) : I \in \mathbb{N},\, \lambda \in \mathbb{R}_+^I,\, \sum_{i=1}^{I} \lambda_i = 1,\, P = \sum_{i=1}^{I} \lambda_i P_i,\, P_i \in C_q(\Gamma) \Big\}$$

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- This parameter is equal to 1 if and only if P is classical

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If $C_q(\Gamma)$ is not closed, such a sequence also exists for $A_q(\cdot)$

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- Extended to optimizing over $C_q^d(\Gamma)$ (Navascués, Feix, Araujo, Vértesi 2015)

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- Our hierarchy is a variant on tracial optimization

SDP hierarchy for lower bounding $A_{qc}(P)$

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$$\mathcal{L}(p) = \sum_i \lambda_i \, \mathrm{Re}(\mathrm{Tr}(p(X(i), Y(i), \psi_i \psi_i^*)))$$

- We have $\mathcal{L}(1) = \sum_i \lambda_i D_{qc}(P_i)$
- Idea: minimize $L(1)$ over linear forms L on $\mathbb{R}\langle x_s^a, y_t^b, z\rangle$ that satisfy certain computationaly tractable properties of $\mathcal{L}$
- Minimization of $L(1)$ used by [Nie 2017] in the commutative setting for the nuclear tensor norm

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- Why SDP? Because $L(p^*p) \geq 0$ is equivalent to $M(L) \succeq 0$, where $M(L)_{u,v} = L(u^*v)$ for all monomials u, v

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Matrix factorization ranks

- $\text{cpsd-rank}(M)$ is the smallest d for which there are Hermitian psd matrices $X_1, \ldots, X_n \in \mathbb{C}^{d \times d}$ with $M_{ij} = \text{Tr}(X_i X_j)$

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- In an earlier paper we have hierarchies for lower bounding matrix factorization ranks, and the hierarchy for $A_q(P)$ is an adaptation of this for (nonsynchronous) quantum correlations

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- Using similar techniques we introduce semidefinite programming hierarchies for the quantum chromatic and quantum stability numbers
- This unifies some of the existing literature; for example, the projective packing number, projective rank, and the tracial rank can be seen as certain steps in the hierarchies

Thank you!

S. Gribling, D. de Laat, M. Laurent, Bounds on entanglement dimensions and quantum graph parameters via noncommutative polynomial optimization, Math. Program., Ser. B (2018), 38 pages