Energy minimization via moment hierarchies

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- As an optimization problem:

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- Hierarchy for packing problems [L.-Vallentin 2014]

This talk

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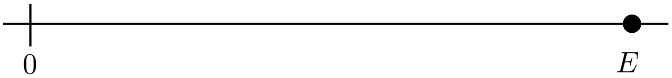
- Towards computations using several types of symmetry reduction

Approach



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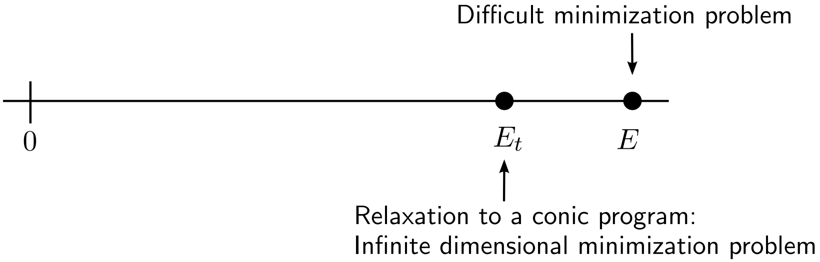
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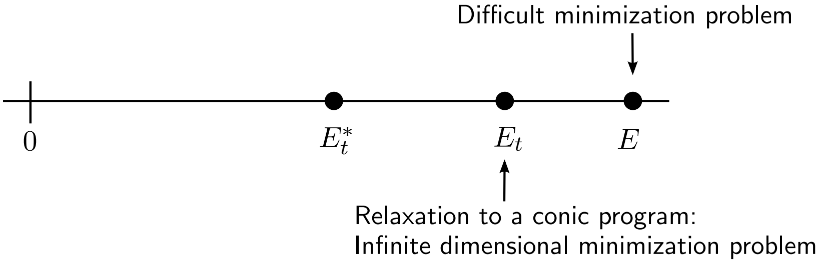

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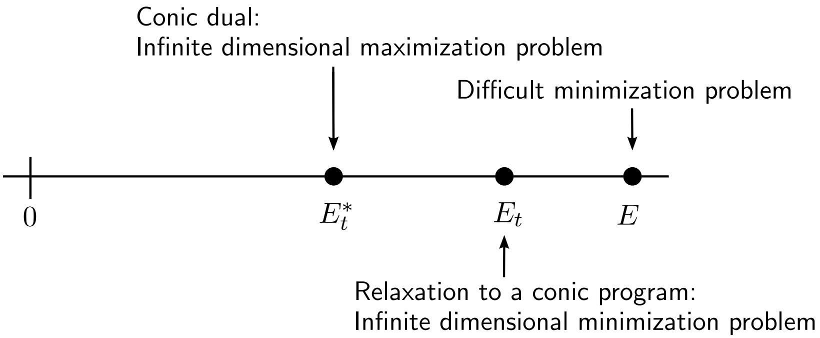

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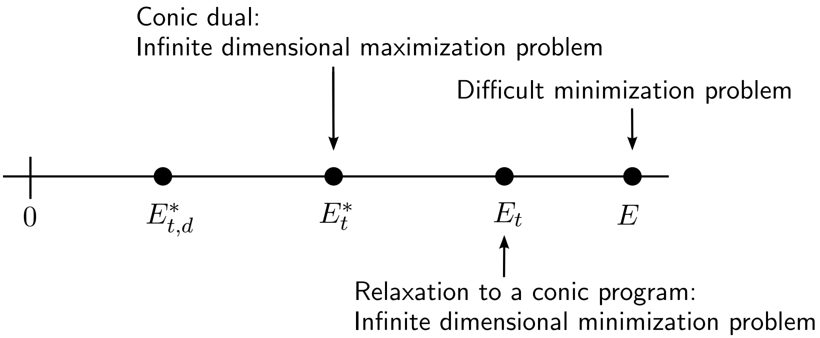
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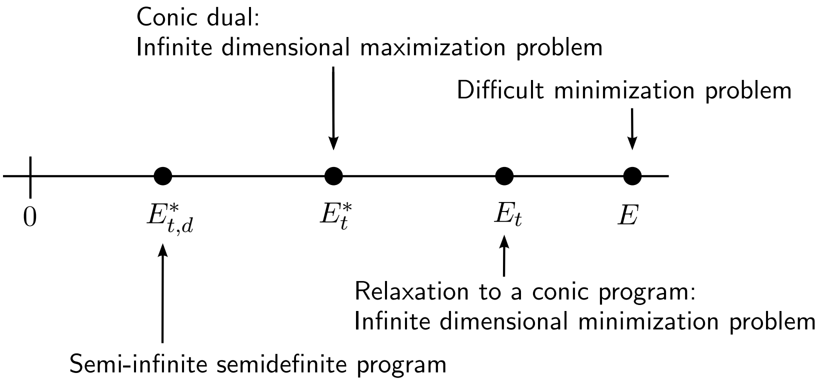
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- $I_{=t}$ gets its topology as a subset of a quotient of V^t

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Theorem (Finite convergence)

We have $E_1 \leq \cdots \leq E_N = E$

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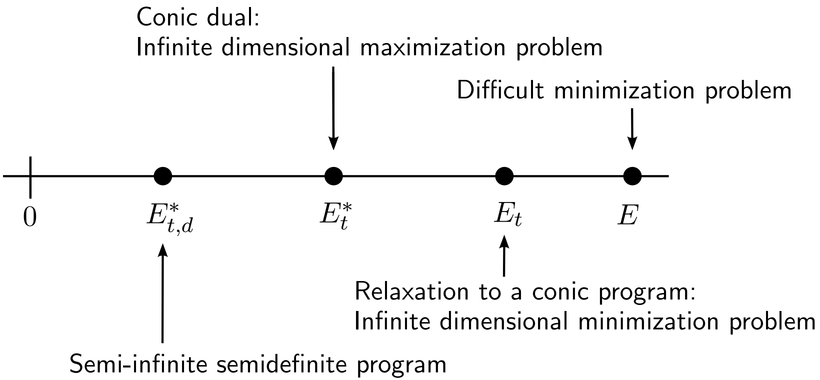
Theorem

Strong duality holds: $E_t = E_t^*$ for each t

Closing the gaps



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Theorem

If V is a compact metric space, then $E_{t,d}^* \to E_t^*$ as $d \to \infty$ for all t

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- If we know how to decompose $\mathcal{C}(V)$ into irreducibles, and how to decompose tensor products of those irreducibles into irreducibles, then we have a symmerty adapted basis of V_t

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- Model nonnegativity constraints as sum of squares constraints using Putinar's theorem from real algebraic geometry

The case $t = 2$ and $V = S^2$

- We know how to these decompositions from the quantum mechanics literature (angular momentum coupling): use Clebsch-Gordan coefficients
- The affine constraints in $E_{t,d}^*$ are nonnegativity constraints of a polynomial $p \in \mathbb{R}[x_1, \ldots, x_4]$, where each x_i is a vector of 3 variables (the coefficients of these polynomials depend on the entries in the block diagonalization of F)
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- Model nonnegativity constraints as sum of squares constraints using Putinar's theorem from real algebraic geometry
- A sum of squares polynomial s can be written as $s(x) = v(x)^{\mathsf{T}} Q v(x)$, where Q is a positive semidefinite matrix and $v(x)$ a vector containing all monomials up to some degree

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- Using a reduction to 3 variables using trigonometric polynomials we compute that $E_2 = E$ (up to solver precision)

Thank you!