A semidefinite programming hierarchy for geometric packing problems

David de Laat (TU Delft)
Joint work with Frank Vallentin (Universität zu Köln)

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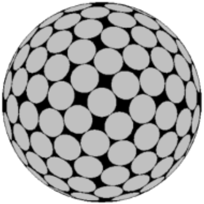
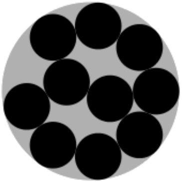
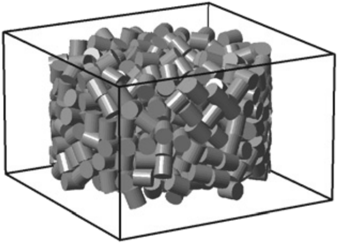
Packing problems in discrete geometry







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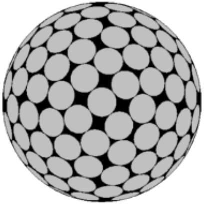 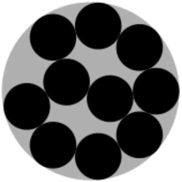 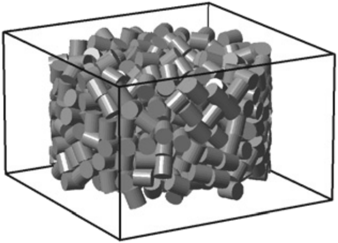

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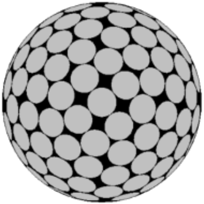
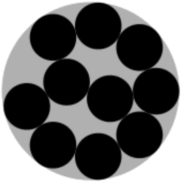
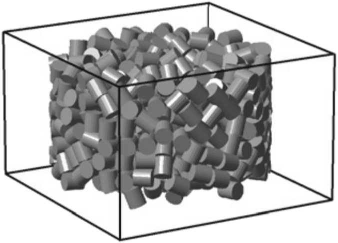

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Spherical cap packings

What is the maximum number of spherical caps of size t in S^{n-1} such that no two caps intersect in their interiors?
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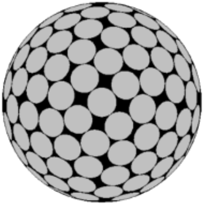
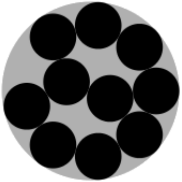
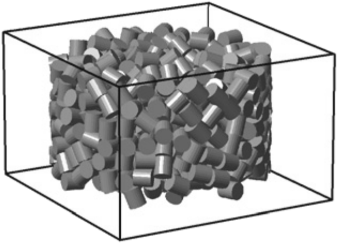

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- Independent sets correspond to valid packings

Upper bounds for the independence number of finite graphs

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- Lasserre hierarchy for the independent set problem (Laurent, 2003)

The Lasserre hierarchy for finite graphs

$$\text{las}_t(G) = \max\left\{ \qquad : y \in \mathbb{R}^{I_{2t}}_{\geq 0}, \qquad , \qquad \right\}$$

- I_t is the set of independent sets of cardinality at most t

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$$M_t(y)_{J,J'} = \begin{cases} y_{J \cup J'} & \text{if } J \cup J' \in I_{2t}, \\ 0 & \text{otherwise} \end{cases}$$

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- $\vartheta'(G) = \mathrm{las}_1(G) \geq \mathrm{las}_2(G) \geq \ldots \geq \mathrm{las}_{\alpha(G)}(G) = \alpha(G)$

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- This talk: Generalize the Lasserre hierarchy to infinite graphs; finite convergence

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- These graphs have finite independence number

Generalization for compact topological packing graphs

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- I_t gets its topology as a subset of $\operatorname{sub}_t(V) \cup \{\emptyset\}$

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- A function $K \in \mathcal{C}(I_t \times I_t)_{\mathsf{sym}}$ is a *positive definite kernel* if

$$(K(J_i, J_j))_{i,j=1}^m \succeq 0 \quad \text{for all} \quad m \in \mathbb{N} \quad \text{and} \quad J_1, \ldots, J_m \in I_t$$

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- Cone of positive definite measures:

$$M(I_t \times I_t)_{\succeq 0} = \{\mu \in M(I_t \times I_t)_{\mathrm{sym}} : \mu(K) \geq 0 \text{ for all } K \in \mathcal{C}(I_t \times I_t)_{\succeq 0}\},$$

where $\mu(K) = \int K(J, J')\, d\mu(J, J')$

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- The adjoint: $A_t^*\colon M(I_{2t}) \to M(I_t \times I_t)_{\mathsf{sym}}$

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- The conic duals are given by

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Theorem

Strong duality holds: for all $t \in \mathbb{N}$,

- $\mathrm{las}_t(G) = \mathrm{las}_t(G)^*$
- if $\mathrm{las}_t(G) < \infty$, then the optimum in $\mathrm{las}_t(G)$ is attained

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  is closed in $M(I_t \times I_t)_{\mathrm{sym}} \times \mathbb{R}$
- Minkowski difference: $K = K_1 - K_2$
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  - $K_2 = \big\{(\mu, 0) : \mu \in M(I_t \times I_t)_{\succeq 0}\big\}$
- By a theorem of Klee and Dieudonné $K_1 - K_2$ is closed when
  1. $K_1 \cap K_2 = \{0\}$
  2. K_1 and K_2 are closed
  3. K_1 is locally compact

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Suppose G is a compact topological packing graph. Then,

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- Main part of the proof: show that σ is a probability measure
- $\operatorname{las}_{\alpha(G)}(G) = \max\{\int |S| \, d\sigma(S) : \sigma \in \mathcal{P}(I_{\alpha(G)})\} = \alpha(G)$

Thank you

D. de Laat, F. Vallentin, *A semidefinite programming hierarchy for packing problems in discrete geometry*, in preparation.

Image credit:
`http://www.buddenbooks.com/jb/images/150a5.gif`
`http://en.wikipedia.org/wiki/File:Disk_pack10.svg`
W. Zhang, K.E. Thompson, A.H. Reed, L. Beenken, *Relationship between packing structure and porosity in fixed beds of equilateral cylindrical particles*, Chemical Engineering Science **61** (2006), 8060–8074.