Using (noncommutative) polynomial optimization to bound matrix factorization ranks

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CWI

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Goal: Find lower bounds for matrix factorization ranks

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- Combine proofs from above refs and [Paulsen–Severini–Stahlke–Todorov–Winter 2016]

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Commutative polynomial optimization is used by Nie for testing membership in the CP cone and computing tensor nuclear norms

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Localizing conditions: $L_X(p^*(\sqrt{A_{ii}} x_i - x_i^2)p) \geq 0$

Truncate to obtain a semidefinite programming hierarchy

$\mathbb{R}\langle x_1, \ldots, x_n \rangle_{2t}$ noncommututative polynomials with $\deg \leq 2t$

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$\xi_*^{\mathrm{cpsd}}(A)$ is $\xi_\infty^{\mathrm{cpsd}}(A)$ with the extra constraint $\mathrm{rank}(M(L)) < \infty$

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- We have $\xi_t^{\mathrm{cpsd}}(A) \to \xi_{\infty}^{\mathrm{cpsd}}(A)$, and if $\xi_t^{\mathrm{cpsd}}(A)$ admits a flat optimal solution, then $\xi_t^{\mathrm{cpsd}}(A) = \xi_{*}^{\mathrm{cpsd}}(A)$

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$$\xi_*^{\mathrm{cpsd}}(A) = \inf\Big\{ \sum_{m=1}^{M} d_m \cdot \max_{i\in[n]} \frac{\|X_i^m\|^2}{A_{ii}} : M \in \mathbb{N},\ d_1, \ldots, d_M \in \mathbb{N},\ X_i^m \in \mathcal{H}_+^{d_m} \text{ for } i \in [n], m \in [M],\ A = \mathrm{Gram}\Big(\bigoplus_{m=1}^{M} X_1^m, \ldots, \bigoplus_{m=1}^{M} X_n^m\Big)\Big\}.$$

Lower bound [Prakash–Sikora–Varvitsiotis–Wei 2016]:

$$\frac{\left(\sum_{i=1}^{n} \sqrt{A_{ii}}\right)^2}{\sum_{i,j=1}^{n} A_{ij}} \leq \mathrm{cpsd\text{-}rank}(A)$$

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Sharp for the matrix $A \in \mathbb{R}^{5\times 5}$ given by $A_{ij} = \cos\big(4\pi/5(i-j)\big)^2$

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$$A = \begin{pmatrix} 1 & 1/2 & 0 & 0 & 1/2 \\ 1/2 & 1 & 1/2 & 0 & 0 \\ 0 & 1/2 & 1 & 1/2 & 0 \\ 0 & 0 & 1/2 & 1 & 1/2 \\ 1/2 & 0 & 0 & 1/2 & 1 \end{pmatrix}$$

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The completely positive rank (cp-rank)

Fawzi and Parrilo (2014) give this SDP to lower bound $\mathrm{cp\text{-}rank}(A)$:

$$\begin{aligned}
\tau_{\mathrm{cp}}^{\mathrm{sos}}(A) = \inf\Big\{ &\alpha : \alpha \in \mathbb{R},\, X \in \mathbb{R}^{n^2 \times n^2}, \\
&\begin{pmatrix} \alpha & \mathrm{vec}(A)^\mathsf{T} \\ \mathrm{vec}(A) & X \end{pmatrix} \succeq 0, \\
&X_{(i,j),(i,j)} \leq A_{ij}^2 \quad \text{for} \quad 1 \leq i, j \leq n, \\
&X_{(i,j),(k,l)} = X_{(i,l),(k,j)} \quad \text{for} \quad 1 \leq i < k \leq n,\ 1 \leq j < l \leq n, \\
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They derive $\tau_{\mathrm{cp}}^{\mathrm{sos}}(A)$ as an SDP relaxation of

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$\tau_{\mathrm{cp}}(A)$ is at least the rank of A and the fractional edge-clique cover number of the support graph of A

Adapting our hierarchy for the cp-rank

Suppose $A_{ij} = v_i^\mathsf{T} v_j$ for $v_1, \ldots, v_n \in \mathbb{R}_+^d$

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This gives a (doubly indexed) sequence of finite semidefinite programs converging asymptotically to $\tau_{\mathrm{cp}}(A)$

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- Implement this constraint more efficiently by exploiting symmetry:

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Then $\xi_{2,+}^{\mathrm{cp}}(A)$ is a more efficient strengthening of $\tau_{\mathrm{cp}}^{\mathrm{sos}}(A)$

The nonnegative rank

The nonnegative rank $\mathrm{rank}_+(A)$ is the smallest d for which there are vectors $u_1, \ldots, u_n, v_1, \ldots, v_n \in \mathbb{R}^d_+$ such that $A_{ij} = u_i^\mathsf{T} v_j$

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Fawzi and Parrilo (2014) define relaxations
$\tau_+^{\mathrm{sos}}(A) \leq \tau_+(A) \leq \mathrm{rank}_+(A)$

The nonnegative rank

The nonnegative rank $\mathrm{rank}_+(A)$ is the smallest d for which there are vectors $u_1, \ldots, u_n, v_1, \ldots, v_n \in \mathbb{R}^d_+$ such that $A_{ij} = u_i^\mathsf{T} v_j$

The nonnegative rank of the slack matrix of a polytope gives the extension complexity of the polytope [Yannakakis 1991]

Fawzi and Parrilo (2014) define relaxations
$\tau_+^{\mathrm{sos}}(A) \leq \tau_+(A) \leq \mathrm{rank}_+(A)$

For $A \in \mathbb{R}^{m \times n}_+$ there are positive semidefinite diagonal matrices $X_1, \ldots, X_{m+n}$ with $A_{ij} = \mathrm{Tr}(X_i X_{m+j})$ and $\lambda_{\max}(X_i)^2 \leq \max_{i,j} A_{ij}$

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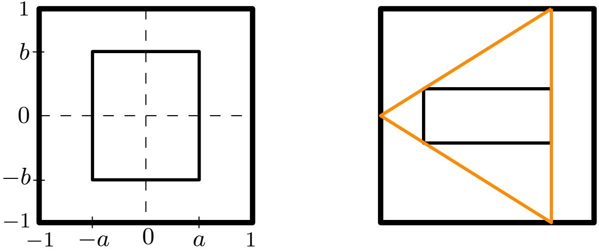
Going back to tracial optimization we can adapt this to the psd-rank – still work in progress

Nested rectangle problem [Fawzi–Parrilo, 2016]:



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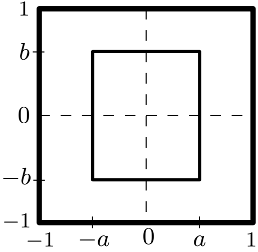

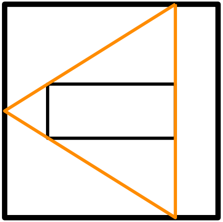

Such a triangle exists if and only if

$$\mathrm{rank}_+\left(\begin{pmatrix} 1-a & 1+a & 1+a & 1-a \\ 1+a & 1-a & 1-a & 1+a \\ 1-b & 1-b & 1+b & 1+b \\ 1+b & 1+b & 1-b & 1-b \end{pmatrix} \right) \leq 3$$

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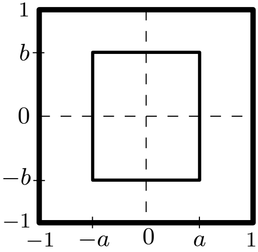

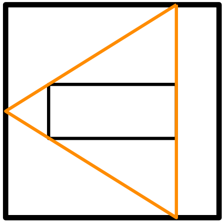

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$$\operatorname{rank}_+\left(\begin{pmatrix} 1-a & 1+a & 1+a & 1-a \\ 1+a & 1-a & 1-a & 1+a \\ 1-b & 1-b & 1+b & 1+b \\ 1+b & 1+b & 1-b & 1-b \end{pmatrix} \right) \leq 3$$

In fact, such a triangle exists if and only if $(1+a)(1+b) \leq 2$

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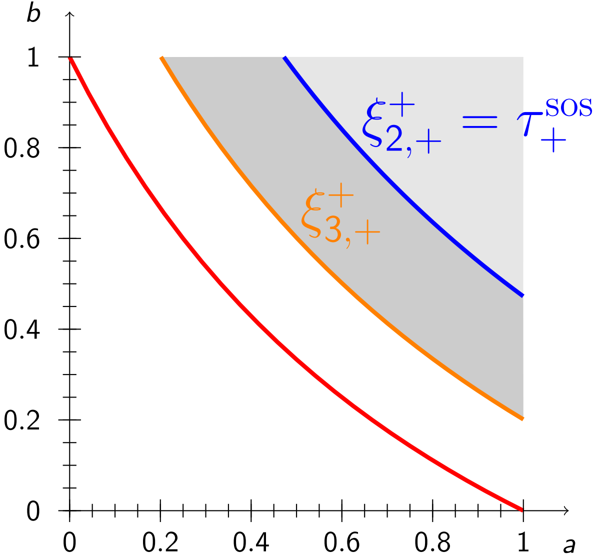

Thank you!