Moment methods in energy minimization

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Delft University of Technology
(Joint with Fernando Oliveira and Frank Vallentin)

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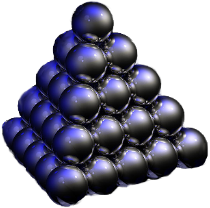
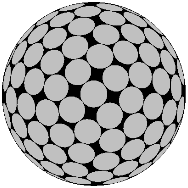
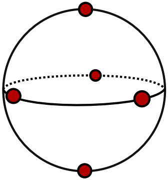
Packing and energy minimization



Sphere packing
Kepler conjecture (1611)



Spherical cap packing
Tammes problem (1930)



Energy minimization
Thomson problem (1904)

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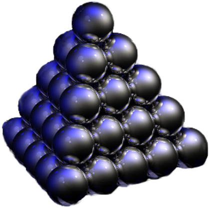

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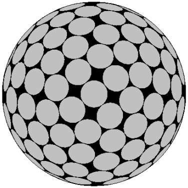

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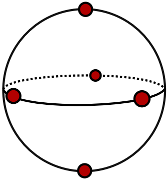

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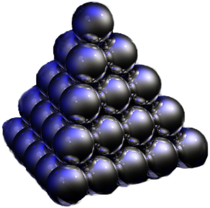

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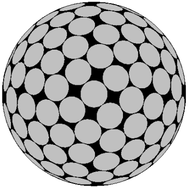

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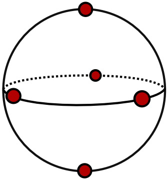

Energy minimization
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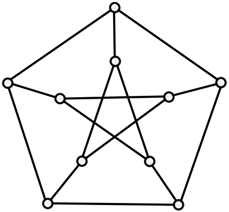
- This talk: Methods to find obstructions

The maximum independent set problem



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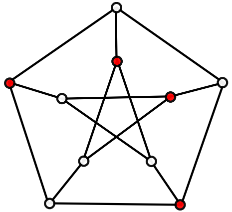

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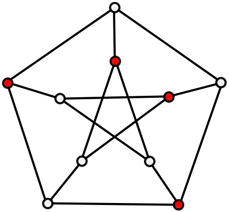

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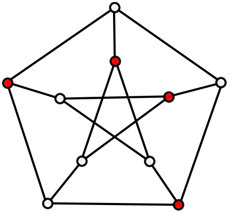

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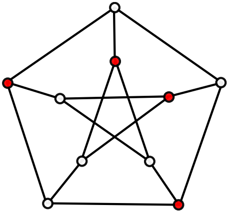

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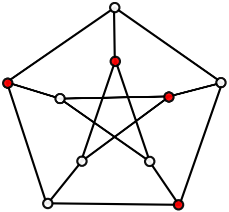

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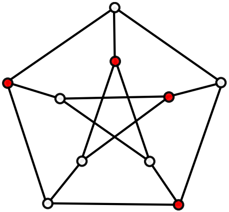

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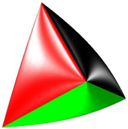
3×3 positive semidefinite matrices with unit diagonal:



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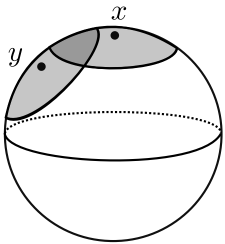

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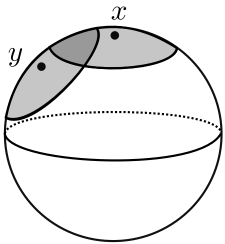

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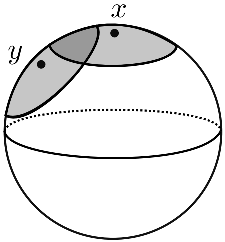

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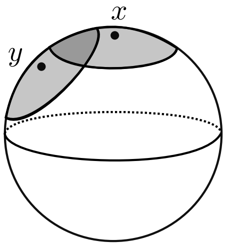

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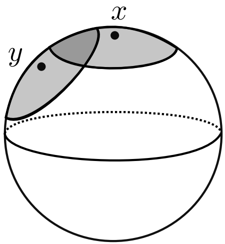

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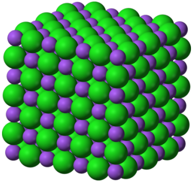
- For this problem this reduces to the Delsarte LP bound

New bounds for binary packings



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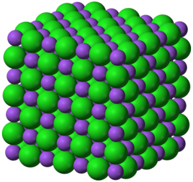

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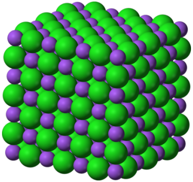

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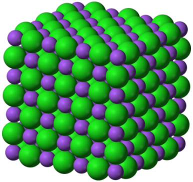

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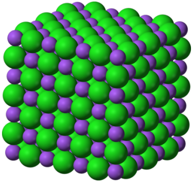

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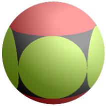

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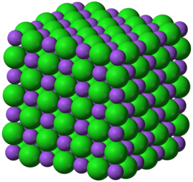

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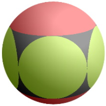

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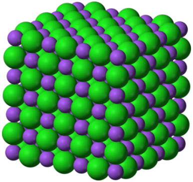

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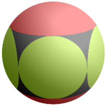

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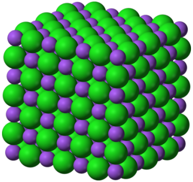

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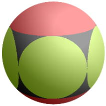

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- Question 2: Can we obtain arbitrarily good bounds?

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$$E = \min_{S \in I_{=N}} \sum_{P \subseteq S} h(P)$$

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- A measure $\lambda \in \mathcal{M}(I_{2t})$ is a *moment measure* if

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- Positive semidefinite form $\langle f, g \rangle = A_t^* \lambda(f \otimes g)$ on $\mathcal{C}(I_t)$
- Define $\mathcal{N}_t(\lambda) = \{f \in \mathcal{C}(I_t) : \langle f, f \rangle = 0\}$
- If $\lambda \in \mathcal{M}(I_{2t})$ is a moment measure and

$$\mathcal{C}(I_t) = \mathcal{C}(I_{t-1}) + \mathcal{N}_t(\lambda),$$

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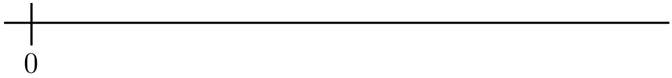
Computations using the dual hierarchy



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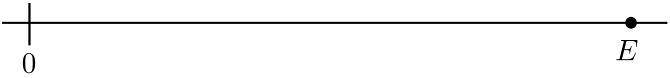

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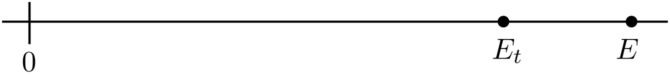

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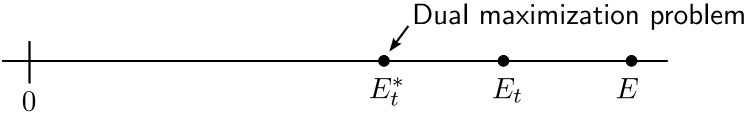

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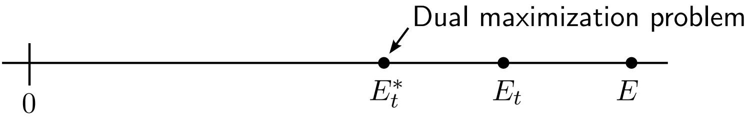

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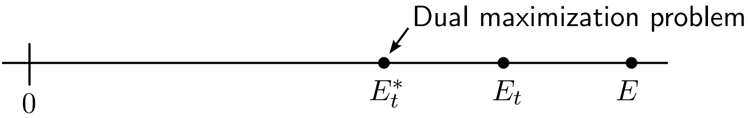

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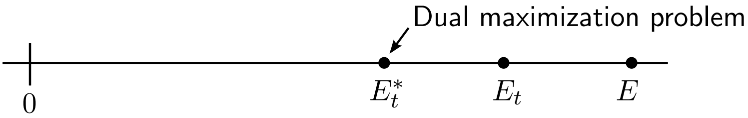

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- Idea: Optimize over truncated Fourier series of K

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- We give a symmetrized version of Putinar's theorem to exploit the S_N symmetry in the particles

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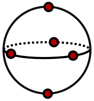

Thank you!

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- D. de Laat, F. Vallentin, **A semidefinite programming hierarchy for packing problems in discrete geometry**, Math. Program., Ser. B 151 (2015), 529-553.
- D. de Laat, F.M. Oliveira, F. Vallentin, **Upper bounds for packings of spheres of several radii**, Forum Math. Sigma 2 (2014), e23 (42 pages).

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