A semidefinite programming hierarchy for geometric packing problems

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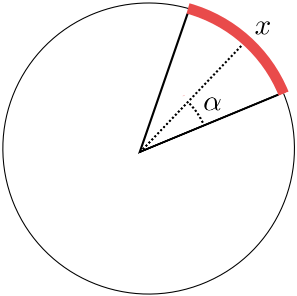
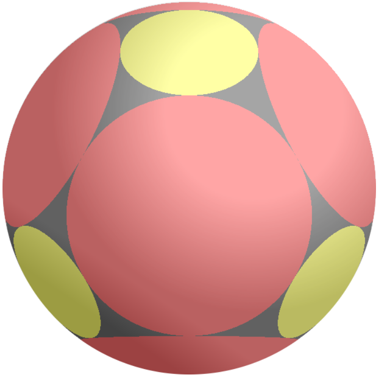
Joint work with Fernando M. de Oliveira Filho and Frank Vallentin

DIAMANT Symposium – November 2012

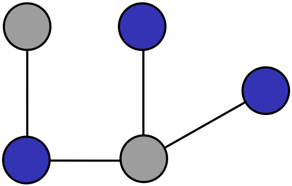
Polydisperse spherical cap packings

How can one pack spherical caps of sizes $\alpha_1, \ldots, \alpha_N$ on the unit sphere as densely as possible?





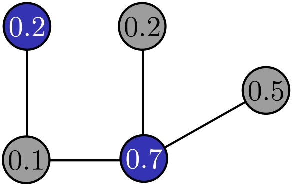
Maximal stable set problem



Simple graph G

Stability number: $\alpha(G) = 3$

Maximal weighted stable set problem



Simple weighted graph G

Weighted stability number: $\alpha_w(G) = 0.9$

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$\alpha_w(G)$ gives the optimal packing density

The theta number for the spherical cap packing graph

$$\vartheta_w(G) = \inf M\colon\ K - \sqrt{w} \otimes \sqrt{w} \in \mathcal{C}(V \times V)_{\succeq 0}\ ,$$
$$K(u,u) \leq M \text{ for all } u \in V,$$
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A kernel $K \in \mathcal{C}(V \times V)_{\succeq 0}^{O(n)}$ is of the form

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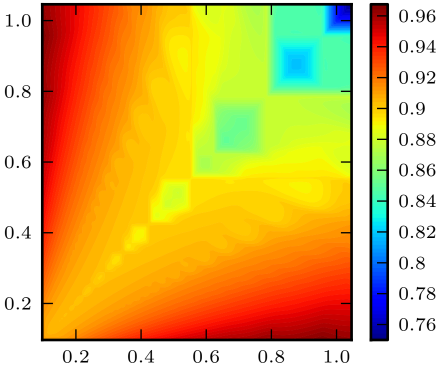
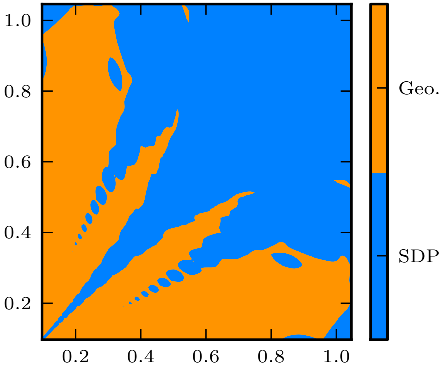
- Use a sums of squares characterization

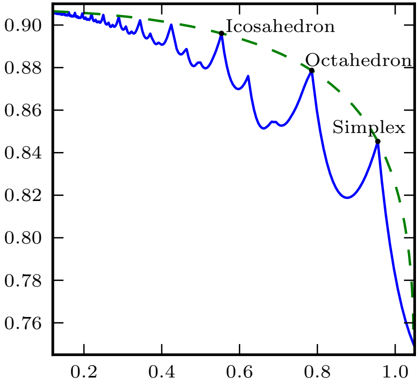
Binary spherical cap packings on the 2-sphere



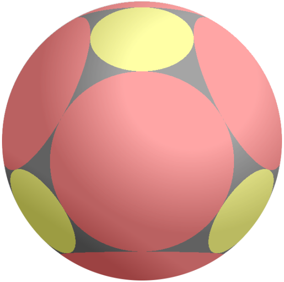
SDP bound / Geometric bound (Florian 2001)



Spherical codes on the 2-sphere



The truncated octahedron packing

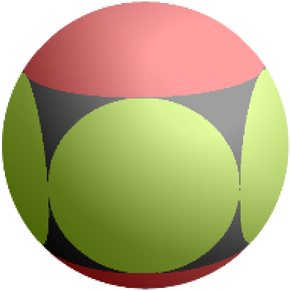


This packing is maximal:

- it has density $0.9056\ldots$
- the semidefinite programming bound is $0.9079\ldots$
- the next packing (4 big caps, 19 small caps) would have density $0.9103\ldots$

Packings associated to the n-prism

- The geometric bound is sharp for $n \geq 6$
- For $n = 5$ there is a geometrical proof (Florian, Heppes 1999)
- The semidefinite programming bound is sharp for $n = 5$



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- Example: graphs where the vertex set is a compact metric space such that x and y are adjacent if $d(x,y) \in (0,\delta)$

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- For $2t \geq \alpha(G)$ we have

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Thank you!

D. de Laat, F.M. de Oliveira Filho, F. Vallentin, *Upper bounds for packings of spheres of several radii*, arXiv:1206.2608, (2012), 31 pages.

D. de Laat, F. Vallentin, *A semidefinite programming hierarchy for geometric packing problems*, in preparation.