New bounds for spherical finite distance sets

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Packing problem 1: Independent sets in finite graphs

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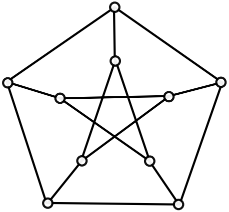

Example: the Petersen graph

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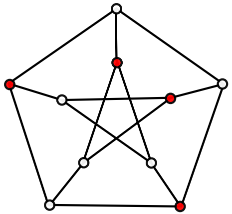

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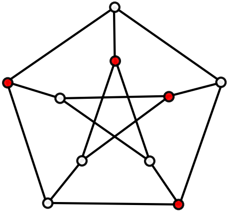

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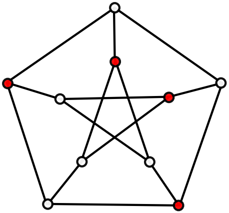

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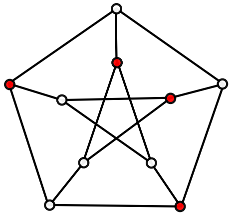

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- Can be computed through semidefinite programming (SDP)

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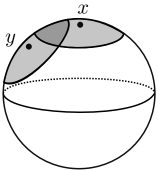

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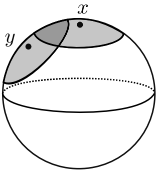

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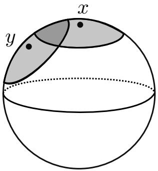

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- [Bachoc-Nebe-Oliveira-Vallentin 2009] showed the Delsarte bound can be interpreted as the ϑ-number of this graph

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- $|I/\Gamma| < \infty$ follows from the fact that any two isometric sets in $\mathbb{R}^n$ are related by an isometry of $\mathbb{R}^n$

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- The Delsarte 2-point bound and Bachoc-Vallentin 3-point bound have been studied extensively in the context of spherical finite distance graphs and equiangular lines [Delsarte, Goethals, Seidel, Barg, Yu, King, Tang, Glazyrin]

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- $T(\gamma x, \gamma y, \gamma Q) = T(x,y,Q)$ for all $\gamma \in \Gamma$

Symmetry reduction

Lemma (L-Machado-Oliveira-Vallentin 2018)
If I_{k-2}/Γ is finite, then we have the homeomorphism

$$\coprod_{R \in \mathcal{R}_{k-2}} V^2/\mathrm{Stab}_\Gamma(R) \simeq (V^2 \times I_{k-2})/\Gamma,$$

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Corollary
If I_{k-2}/Γ is finite, then we have the isomorphism

$$\Psi\colon \bigoplus_{R \in \mathcal{R}_{k-2}} \mathcal{C}(V^2)^{\mathrm{Stab}_\Gamma(R)} \to \mathcal{C}(V^2 \times I_{k-2})^\Gamma$$

given by $\Psi(K)(x, y, Q) = K_{\gamma_Q^{-1} Q}(\gamma_Q^{-1} x, \gamma_Q^{-1} y)$

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Theorem (Musin 2014 / Nonorthogonal extension LMOV 2018)
Every

$$K \in \mathcal{C}(S^{n-1} \times S^{n-1})_{\succeq 0}^{\mathrm{Stab}_{O(n)}(\mathrm{span}(R))}$$

can be approximated uniformly by kernels of the form

$$K(x, y) = \sum_{l=0}^{d} \mathrm{trace}(F_l \, Y_l^{n,m}(x \cdot y, L_{A_R} x, L_{A_R} y)),$$

where the matrices F_l are positive semidefinite

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- Currently computations for $k = 4, 5$

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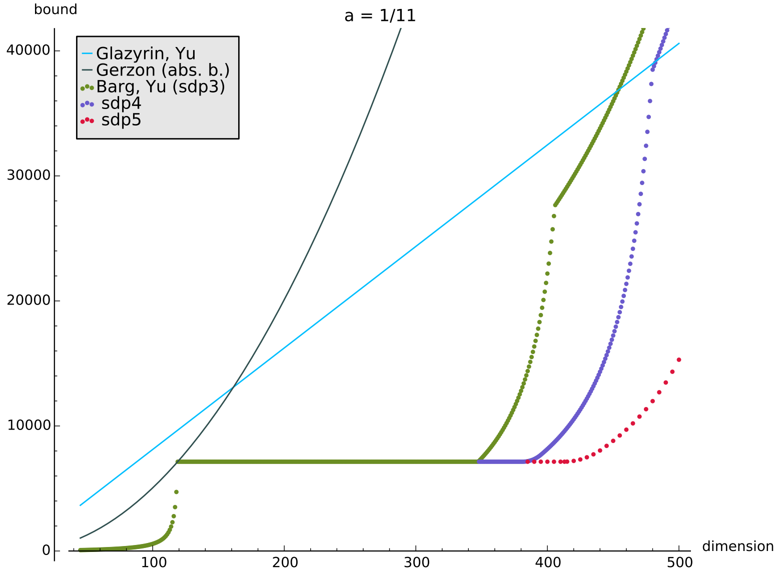
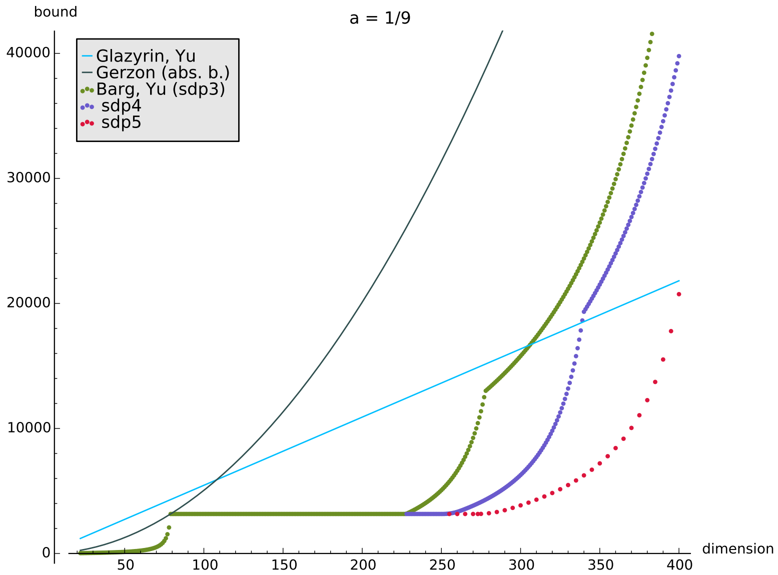


Spherical finite distance graph with $a_1 = a$, $a_2 = -a$



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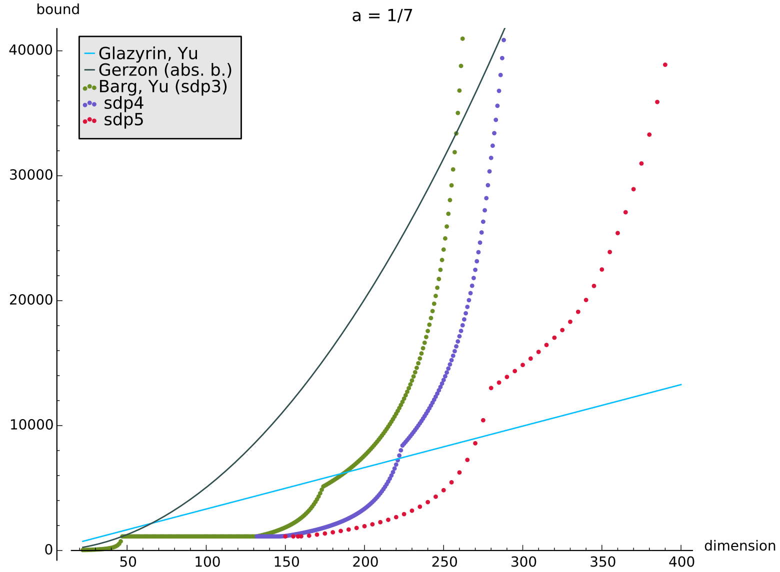

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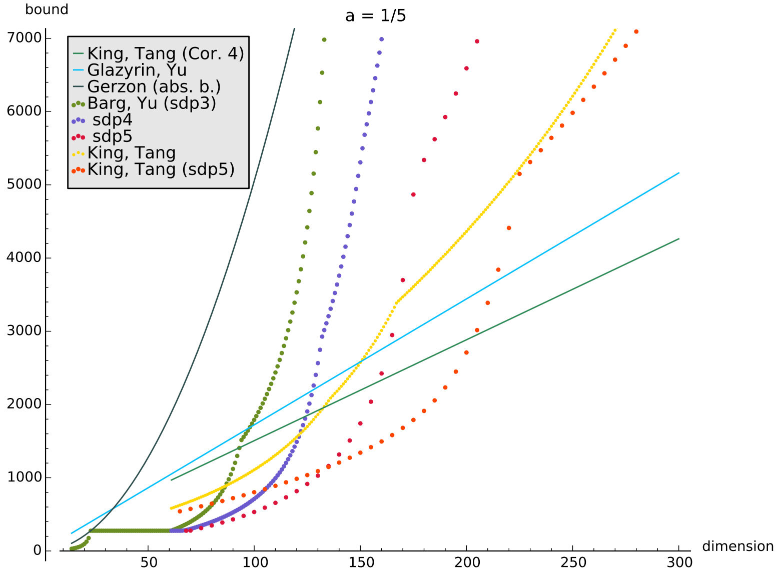

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Theorem

Convergence: $\mathrm{las}_{\alpha(G)}(G)^* = \alpha(G)$

Adaptation of the Lasserre hierarchy for packing

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$$\mathrm{las}_t(G)^* = \inf \Big\{ K(\emptyset, \emptyset) : K \in \mathcal{C}(I_t \times I_t)_{\succeq 0}, \\ A_t K(S) \leq -1_{I_{=1}}(S) \text{ for } S \in I'_{2t} \Big\}$$

$$A_t \colon \mathcal{C}(I_t \times I_t) \to \mathcal{C}(I_{2t}),\ A_t K(S) = \sum_{J, J' \in I_t : J \cup J' = S} K(J, J')$$

- $\alpha(G) \leq \mathrm{las}_t(G)^*$ for all compact topological packing graphs G
- $\mathrm{las}_t(G)^*$ is a $2t$-point bound

Theorem
Convergence: $\mathrm{las}_{\alpha(G)}(G)^* = \alpha(G)$
(The proof uses the primal)

Adaptation to energy minimization (L-2016)

The following optimization problem gives a lower bound on the ground state energy of N particles in V with pair potential f:

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- See Schwartz' talk on Friday for his approach that solves this problem for all s in an interval containing the phase transition

Specialization to finite distance graphs (LMOV 2018)

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- This is like generating all spherical harmonics if you only need the Jacobi polynomials

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- Find the right representations H_π of $O(n)$
- We are essentially interested in

$$H_\pi^{SO(n-i)} \quad \text{for} \quad i = 0, \ldots, t$$

  where π is a representation of $SO(n)$

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"seems to be an act of providence"

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- The implementation is work in progress

Thank you!