Upper bounds for multiple-size spherical cap packings

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Joint work with Fernando de Oliveira Filho and Frank Vallentin

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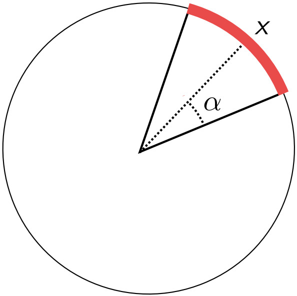
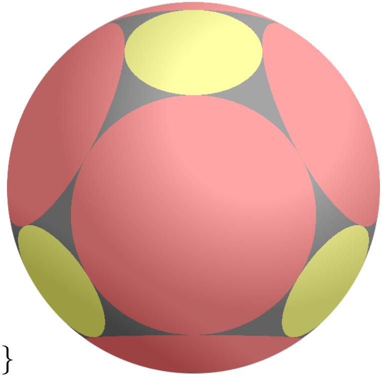
Multiple-size spherical cap packing problem

Given a fixed set $\{\alpha_1, \ldots, \alpha_N\}$ of spherical cap angles:
What is the maximal spherical cap packing density?



$$C(x, \alpha) = \{y \in S^{n-1} : x \cdot y \geq \cos \alpha\}$$



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- Stable sets correspond to packings
- Weighted stability number gives the maximal packing density

The weighted theta number for finite graphs

- Computing the weighted stability number is NP-hard
- The weighted theta (prime) number gives upper bounds:

$$\vartheta'_w(G) = \min \left\{ M : \begin{array}{c} K - \sqrt{w}\sqrt{w}^{\mathsf{T}} \in S^V_{\succeq 0} \\ (K - MI)(u,v) \leq 0 \text{ when } u \not\sim v \end{array} \right\}$$

- This can be computed in polynomial time using semidefinite programming

Infinite graphs

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- Generalization of the theta number:

$$\vartheta'_w(G) = \inf \left\{ M : \begin{array}{c} K - \sqrt{w} \otimes \sqrt{w} \in \mathcal{C}_{\succeq 0}(V \times V) \\ (K - MI)(u, v) \leq 0 \text{ when } u \not\sim v \end{array} \right\}$$

- Infinitely many variables/constraints: how to compute this?

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then $(\overline{K}, M)$ is also feasible, where

$$\overline{K}(u,v) := \int_{O(n)} K(gu, gv)\mu(dg)$$

and where μ is the normalized Haar measure on $O(n)$

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Symmetry reduction:

In the infimum we can restrict to $O(n)$-invariant kernels

Positive invariant kernels

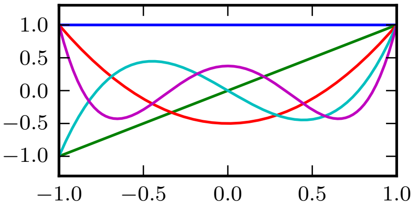
A kernel $K \in \mathcal{C}(V \times V)$ is positive and $O(n)$-invariant if and only if

$$K((x,i),(y,j)) = \sum_{k=0}^{\infty} f_{k,ij} P_k^n(x \cdot y),$$

where $(f_{k,ij})_{1 \leq i,j \leq N} \succeq 0$ for all k
(For $N = 1$ this is a result of Schoenberg)

Jacobi polynomials P_k^n:

Orthogonal with respect to the weight $(1 - s^2)^{(n-3)/2}$ on $[-1, 1]$
Normalized such that $P_k^n(1) = 1$



Simplified program

This gives the following simplified program

$$\vartheta'_w(G) = \inf \left\{ M : \begin{array}{l} \left(f_{0,ij} - \sqrt{w(\alpha_i)}\sqrt{w(\alpha_j)}\right)_{1\le i,j\le N} \succeq 0 \\ \left(f_{k,ij}\right)_{1\le i,j\le N} \succeq 0,\ k = 1,2,\ldots \\ \sum_{k=0}^{\infty} f_{k,ij} P_k^n(u) \le 0,\ -1 \le u \le \cos(\alpha_i + \alpha_j) \\ \sum_{k=0}^{\infty} f_{k,ii} \le M \end{array} \right\}$$

- Replace ∞ by a large number d
- Finitely many variables, but still infinitely many constraints
- Use a sum of squares characterization to simplify further:

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If p is a real even univariate polynomial, then

$$p(x) \geq 0 \text{ for all } x \in [a, b] \Leftrightarrow p(x) = q(x) + (x - a)(b - x)r(x)$$

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$$p(x) \text{ is SOS } \Leftrightarrow p(x) = [x]^T Q[x] \text{ for some } Q \succeq 0$$

$$[x] = (1, x, \ldots, x^d)^{\mathsf{T}}$$

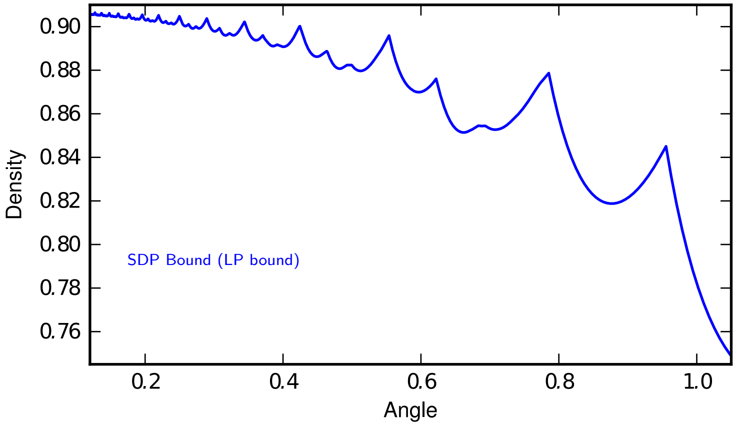
A semidefinite program

This gives the semidefinite program

$$\vartheta'_w(G) = \inf \left\{ M : \begin{array}{l} \left(f_{0,ij} - \sqrt{w(\alpha_i)}\sqrt{w(\alpha_j)}\right)_{1\le i,j\le N} \succeq 0 \\ \left(f_{k,ij}\right)_{1\le i,j\le N} \succeq 0,\ k = 1,\ldots,d \\ Q^{ij} \in S^{d/2+1}_{\succeq 0}, R^{ij} \in S^{d/2}_{\succeq 0},\ 1 \le i,j \le N \\ \Sigma_{k=0}^{d} (P_l^n)_k f_{l,ij} + \langle Q^{ij}, E_l\rangle + \langle R^{ij}, T_{l,ij}\rangle = 0 \\ \Sigma_{k=0}^{d} f_{k,ii} \le M \end{array} \right\}$$

- E_l is the 0/1 matrix with $(E_l)_{i+1,j+1} = 1$ when $i + j = l$
- $T_{l,ij} = \cos(\alpha_i + \alpha_j)E_l + (-1 + \cos(\alpha_i + \alpha_j))E_{l-1} - E_{l-2}$

Single size packings ($n = 3$)



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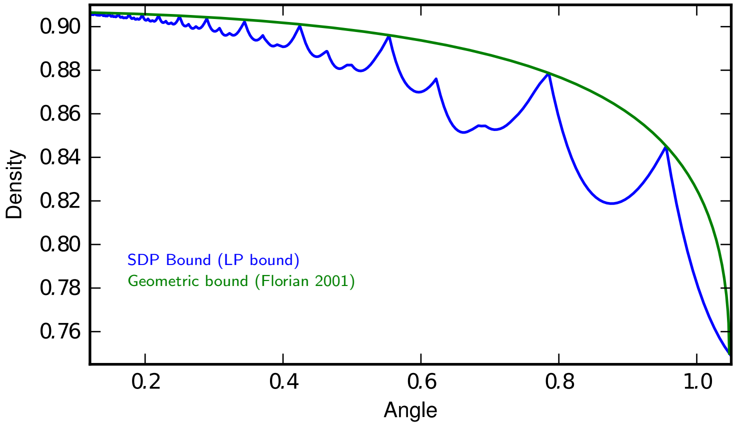

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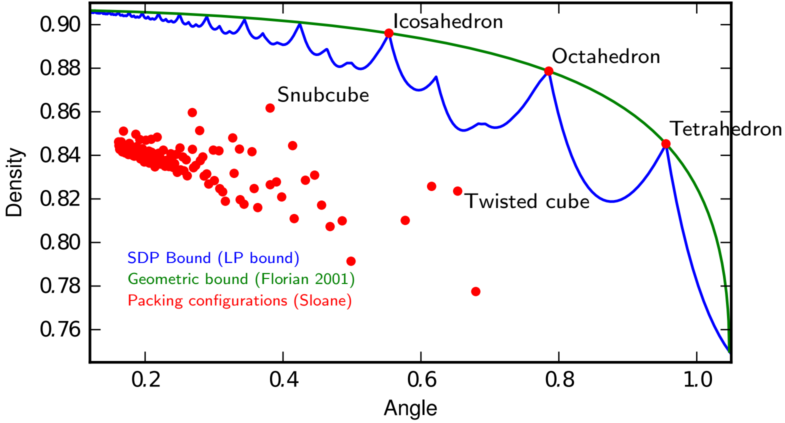

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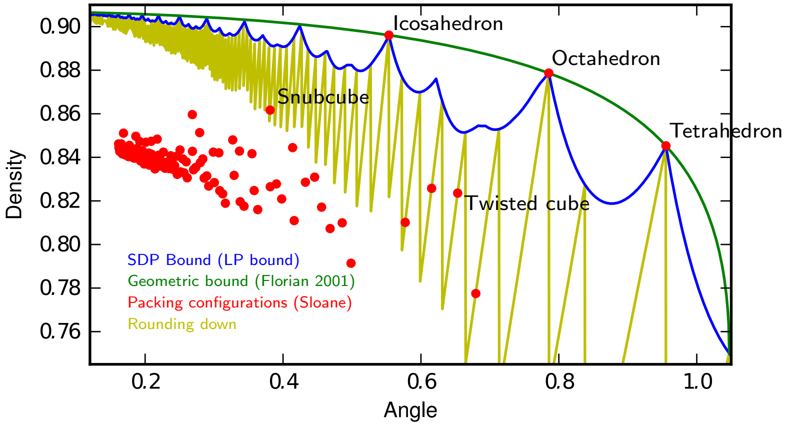
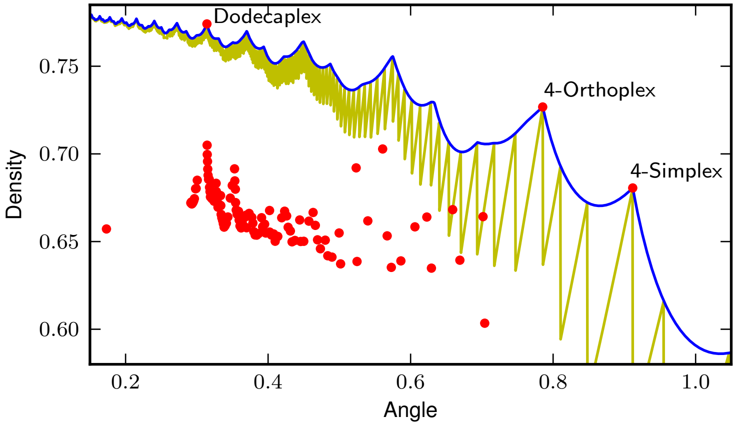
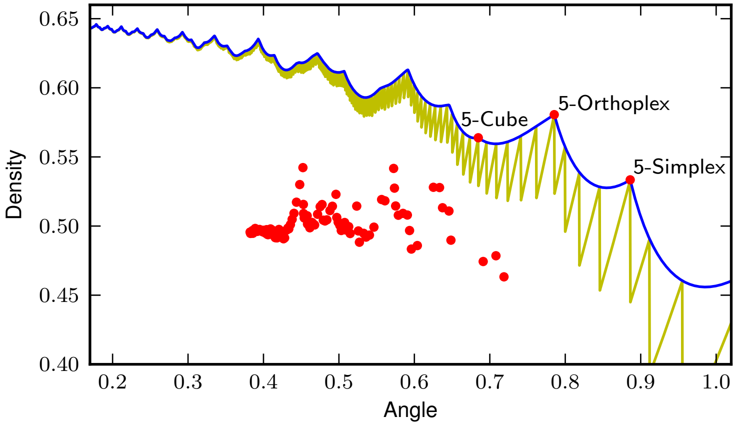


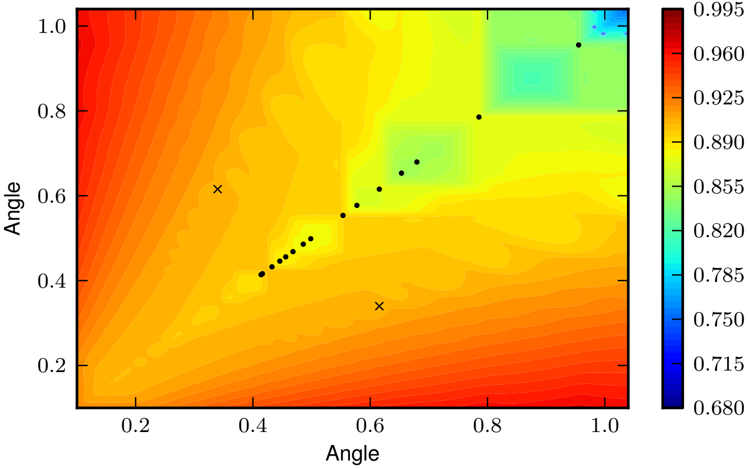
Single size packings for $n = 4$



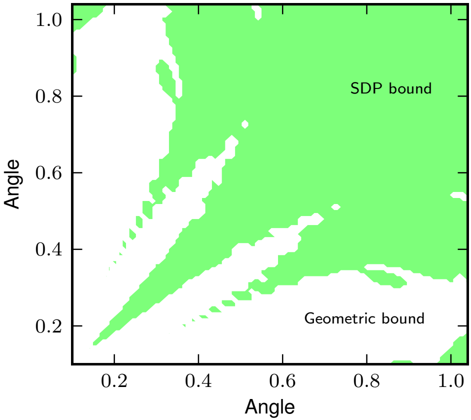
Single size packings for $n = 5$



Binary packings for $n = 3$



SDP bound / Geometric bound



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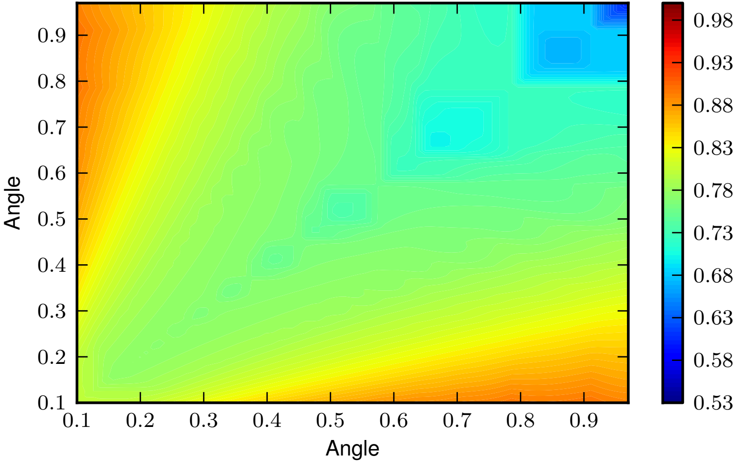

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