Polydisperse spherical cap packings

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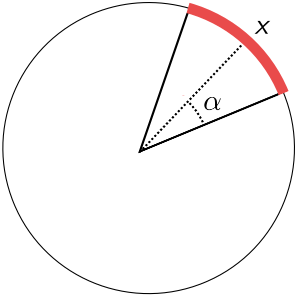
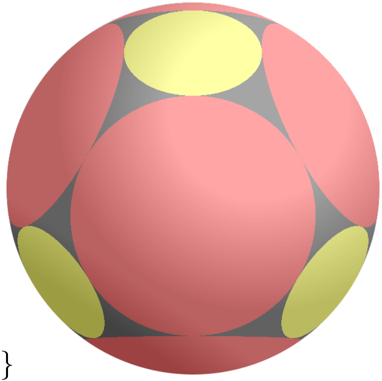
Given a set $\{\alpha_1, \ldots, \alpha_N\}$ of spherical cap angles:
What is the maximal spherical cap packing density?





$C(x, \alpha) = \{y \in S^{n-1} : x \cdot y \geq \cos \alpha\}$

$w(\alpha)$ = normalized cap area of a cap with angle α

The theta number for the packing graph

Packing graph G: $V = S^{n-1} \times \{1, \ldots, N\}$

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The weighted independence number gives the maximal packing density

$$\begin{aligned} \vartheta'_w(G) = \inf M\colon\ & K - \sqrt{w} \otimes \sqrt{w} \in \mathcal{C}(V \times V)_{\succeq 0}, \\ & K(u, u) \leq M \text{ for all } u \in V, \\ & K(u, v) \leq 0 \text{ for all } \{u, v\} \notin E \text{ where } u \neq v. \end{aligned}$$

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By averaging a feasible solution under the group action, we see that we can restrict to $O(n)$ invariant kernels:

Replace $\mathcal{C}(V \times V)_{\succeq 0}$ by $\mathcal{C}(V \times V)_{\succeq 0}^{O(n)}$

The theta number for the packing graph

$$V = S^{n-1}$$

A kernel $K \in \mathcal{C}(V \times V)$ is positive and $O(n)$-invariant if and only if

$$K(x, y) = \sum_{k=0}^{\infty} f_k P_k^n(x \cdot y),$$

where $f_k \geq 0$ for all k (Schoenberg)

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$$K((x, i), (y, j)) = \sum_{k=0}^{\infty} f_{ij,k} P_k^n(x \cdot y),$$

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The theta number program for the packing graph reduces to

$$\begin{aligned} \inf M\colon & (f_{ij,0} - w(\alpha_i)^{1/2} w(\alpha_j)^{1/2})_{i,j=1}^N \succeq 0, \\ & (f_{ij,k})_{i,j=1}^N \succeq 0 \text{ for } k \geq 1, \\ & f_{ij}(u) \leq 0 \text{ whenever } -1 \leq u \leq \cos(\alpha_i + \alpha_j), \\ & f_{ii}(1) \leq M \text{ for all } i = 1, \ldots, N \end{aligned}$$

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If p is a real even univariate polynomial, then

$$p(x) \geq 0 \text{ for all } x \in [a, b] \Leftrightarrow p(x) = q(x) + (x - a)(b - x) r(x)$$

where q and r are SOS polynomials

A direct proof of the upper bounding property

Let $\bigcup_{i=1}^{m} C(x_i, \alpha_{r(i)})$ be a packing, $r\colon \{1, \dots, m\} \to \{1, \dots, N\}$

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$$S = \sum_{k=0}^{\infty} \sum_{i,j=1}^{m} \sqrt{w(\alpha_{r(i)})}\sqrt{w(\alpha_{r(j)})} f_{r(i)r(j),k} P_k^n(x_i \cdot x_j)$$

$$\ge \sum_{i,j=1}^{m} \sqrt{w(\alpha_{r(i)})}\sqrt{w(\alpha_{r(j)})} f_{r(i)r(j),0}$$

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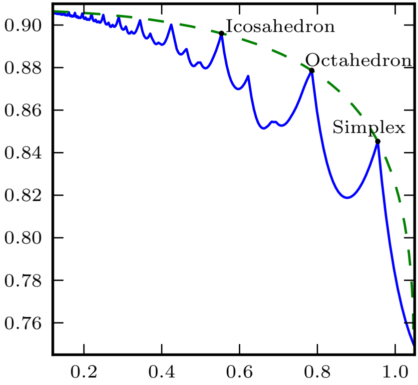
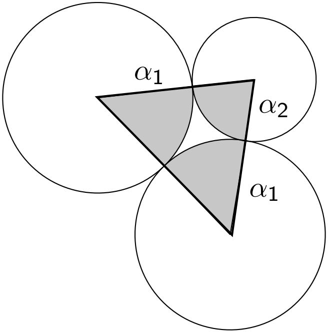
So, $\max\{f_{ii}(N)\colon i = 1,\ldots,N\} \geq \sum_{i=1}^{m} w(\alpha_{r(i)})$. $\square$

Single size packings on the 2-sphere

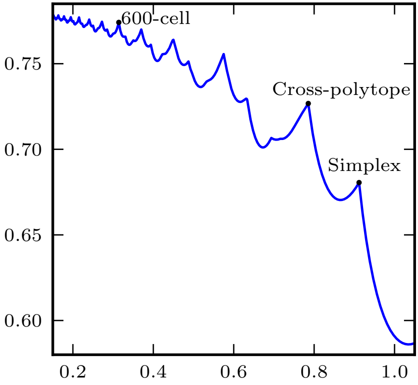


Geometric bound on the 2-sphere (Florian 2001)

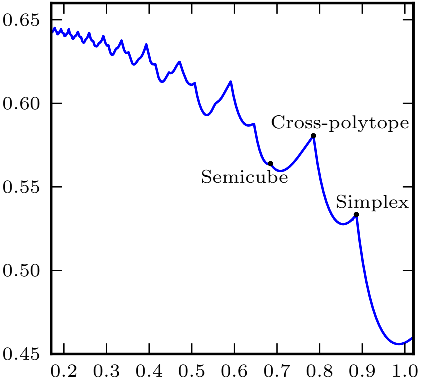


- $D(\alpha_1, \alpha_1, \alpha_2)$ = area of shaded part/area of spherical triangle
- $\max_{1 \leq i \leq j \leq k \leq N} D(\alpha_i, \alpha_j, \alpha_k)$ upper bounds the packing density

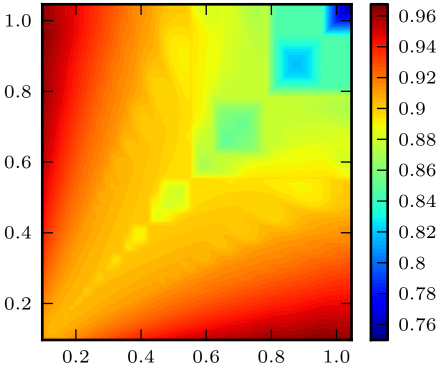
Single size packings on the 4-sphere



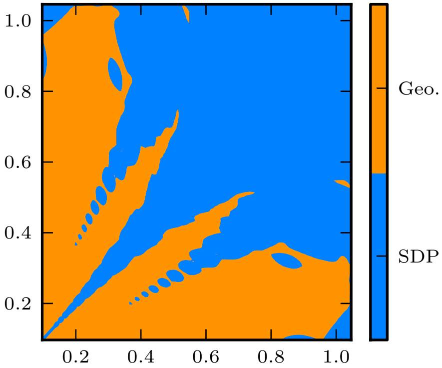
Single size packings on the 5-sphere



Binary packings on the 2-sphere



SDP bound / Geometric bound



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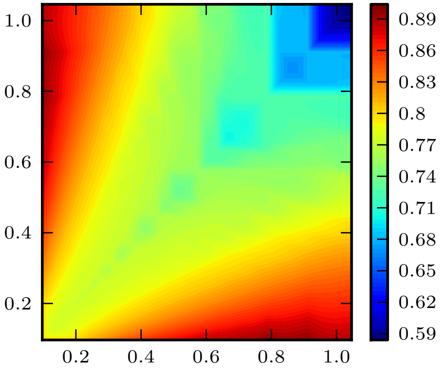

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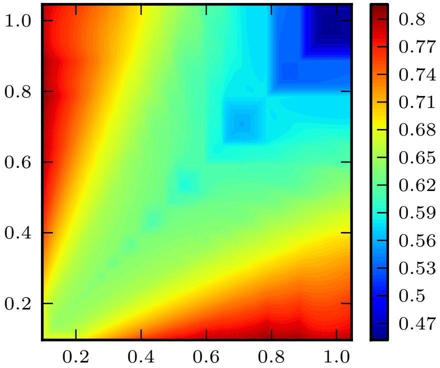
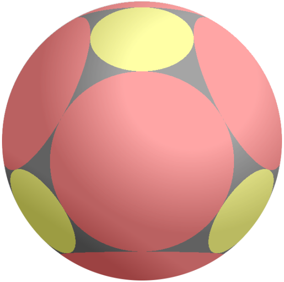


The truncated octahedron packing



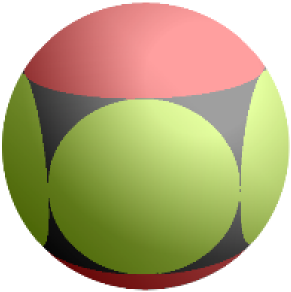
This packing is maximal:

- it has density 0.9056 . . .
- the semidefinite program gives 0.9079 . . .
- the next packing (4 big caps, 19 small caps) would have density 0.9103 . . .

The n-prism packings

Packings associated to the n-prism

- The geometric bound is tight for $n \geq 6$
- For $n = 5$ there is a geometrical proof (Florian, Heppes 1999)
- The numerical solution suggest that the semidefinite programming bound is tight for $n = 5$



The bound is tight for the 5-prism

We need to find functions

$$f_{ij}(u) = \sum_{k=0}^{4} f_{ij,k} P_k^n(u)$$

that satisfy the constraints of the theorem with

$$\max\{f_{11}(1), f_{22}(1)\} = \text{density of the 5-prism packing}$$

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- We obtain 9 linear independent relations on the coefficients

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- We use the fact that $\langle A, B \rangle = 0$ implies $AB = 0$ for positive semidefinite matrices A and B
- We obtain 9 linear independent relations on the coefficients
- By adding two guesses based on the numerical solution we can pick a solution from the remaining one dimensional space

Thank you!