Moment methods in extremal geometry

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Delft University of Technology
(Joint with Fernando Oliveira and Frank Vallentin)

51st Dutch Mathematical Congress
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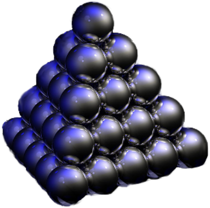
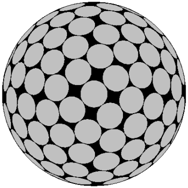
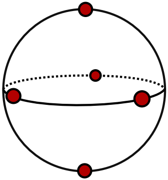
Packing and energy minimization



Sphere packing
Kepler conjecture (1611)



Spherical cap packing
Tammes problem (1930)



Energy minimization
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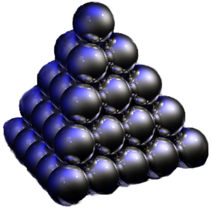

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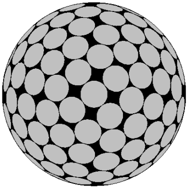

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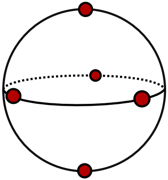

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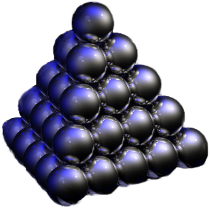

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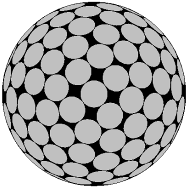

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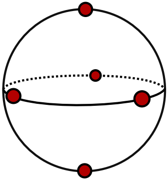

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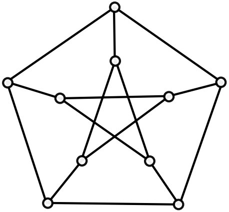
- This talk: Methods to find obstructions

The maximum independent set problem



Example: the Petersen graph

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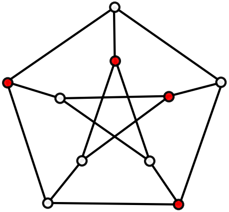

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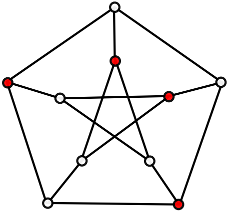

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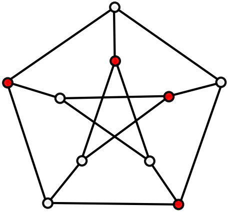

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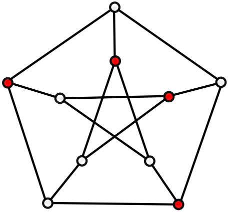

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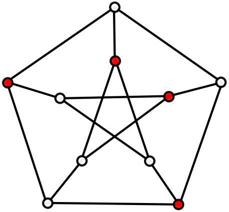

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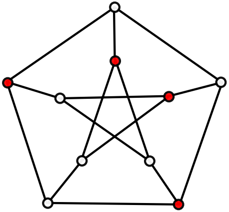

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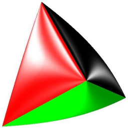
3×3 positive semidefinite matrices with unit diagonal:



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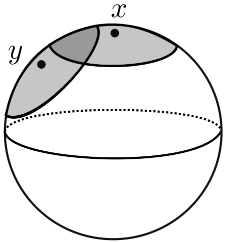

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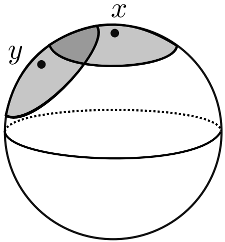

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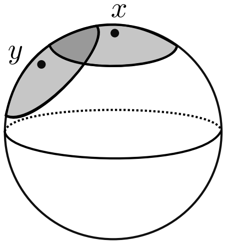

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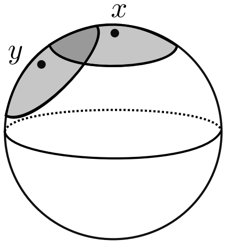

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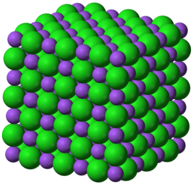
- ϑ generalizes to an infinite dimensional maximization problem
- Use optimization duality, harmonic analysis, and real algebraic geometry to approximate ϑ by a semidefinite program

New bounds for binary packings



Sodium Chloride

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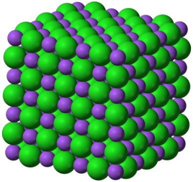

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Density: $79.3\ldots\%$

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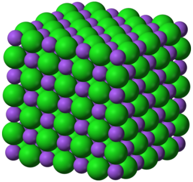

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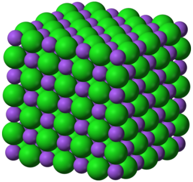

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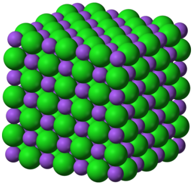

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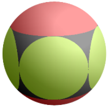

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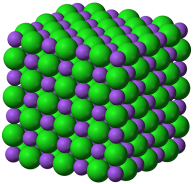

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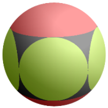

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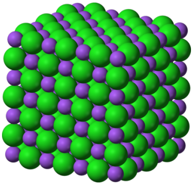

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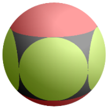

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- Question 2: Can we obtain arbitrarily good bounds?

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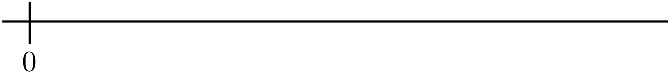
$$E_1 \leq E_2 \leq \cdots \leq E_N = E$$

Computations using the dual



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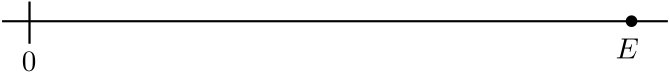

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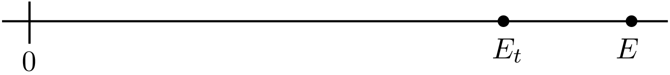

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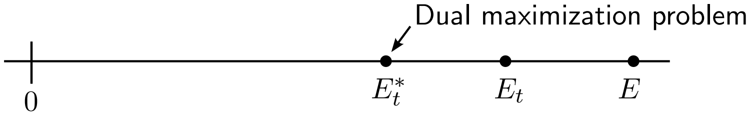

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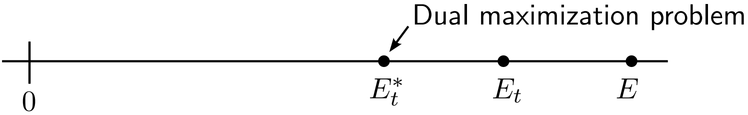

Strong duality holds: $E_t = E_t^*$

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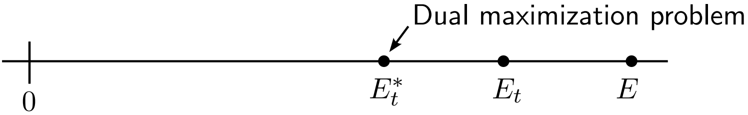

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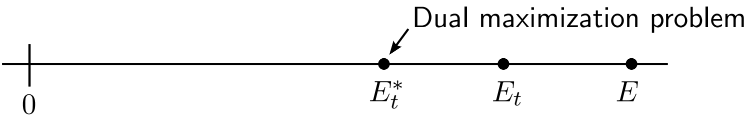

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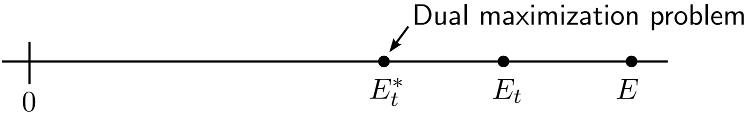

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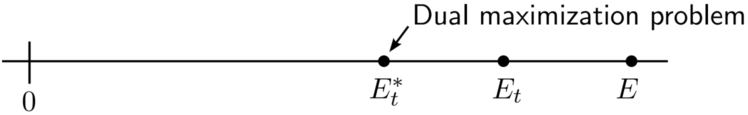
- The Thomson problem has been solved for:
  3 (1912), 4, 6 (1992), 12 (1996), and 5 (2010) particles

Computations using the dual



Strong duality holds: $E_t = E_t^*$

- In E_t^* we optimize over continuous, positive definite kernels

$$K\colon \binom{V}{\leq t} \times \binom{V}{\leq t} \to \mathbb{R}$$

- Optimize over truncated Fourier series of K
- The Thomson problem has been solved for:
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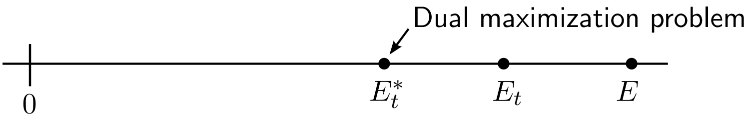

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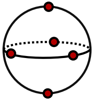
- E_1^* is sharp for 3, 4, 6, and 12 particles
- E_2^* is (almost) sharp for 5 particles



Thank you!

Image credits:
Sphere packing: Grek L
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Sodium Chloride: Ben Mills