A semidefinite programming hierarchy for geometric packing problems

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Joint work with Frank Vallentin

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Geometric packing problems

- Spherical codes (spherical cap packings): What is the largest number of points that one can place on S^{n-1} such that the pairwise inner products are at most t?
- Model geometric packing problems as maximum independent set problems
- $G = (S^{n-1}, x \sim y \text{ if } x \cdot y > t)$

Definition

A *packing graph* is a graph where

- the vertex set is a Hausdorff topological space
- each finite clique is contained in an open clique

- We will consider compact packing graphs

Upper bounds for the max independent set problem

<table>
<tr><th>Finite Graphs</th><th>Infinite Graphs</th></tr>
<tr><td>Delsarte, 1973</td><td>Delsarte, 1977
Kabatiansky, Levenshtein, 1978</td></tr>
<tr><td>Lovász, 1979</td><td rowspan="2">Bachoc, Nebe, de Oliveira, Vallentin, 2009</td></tr>
<tr><td>Schrijver, 1979
McEliece, Rodemich, Rumsey, 1978</td></tr>
<tr><td>Lasserre, 2001</td><td rowspan="2"></td></tr>
<tr><td>Laurent, 2003</td></tr>
</table>

This talk: generalize the Lasserre hierarchy to infinite graphs and prove finite convergence

The Lasserre hierarchy for finite graphs

$$\alpha \leq \max\left\{ \sum_{i=1}^n y_i : \begin{array}{c|ccccccccc|} & \emptyset & 1 & 2 & \cdots & n & \{1,2\} & \cdots & \{3,5\} & \cdots \\ \hline \emptyset & 1 & & & & & & & \vdots & \\ 1 & & y_1 & & & & & & \vdots & \\ 2 & & & y_2 & & & & & \vdots & \\ \vdots & & & & \ddots & & & & \vdots & \\ n & & & & & y_n & & & \vdots & \\ \{1,2\} & & & & & & & & \vdots & \\ \vdots & & & & & & & & \vdots & \\ \{3,4\} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & y_{\{3,4,5\}} & \\ \vdots & & & & & & & & & \end{array} \succeq 0, \quad y_S = 0 \text{ if } S \text{ has an edge} \right\}$$

Finite subset spaces

- $\mathsf{Sub}(V, t)$ is the collection of nonempty subsets of V with at most t elements
- Quotient map:

$$q\colon V^t \to \mathsf{Sub}(V, t), (v_1, \ldots, v_t) \mapsto \{v_1, \ldots, v_t\}$$

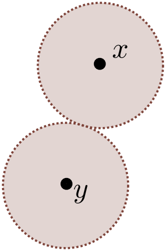
- $\mathsf{Sub}(V, t)$ is a compact Hausdorff space
- $I_t \subseteq \mathsf{Sub}(V, t)$ is the collection of nonempty independent sets with at most t elements
- $V_t = \mathsf{Sub}(V, t) \cup \{\emptyset\}$ is part of the semigroup $(2^V, \cup)$

Finite subset spaces

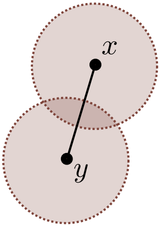
I_t is the collection of nonempty independent sets with $\leq t$ elements

Lemma

I_t is compact



$\{x, y\} \in I_2$



$\{x, y\} \notin I_2$

Finite subset spaces

- If the topology on V comes from a metric, then the topology on Sub(V, t) is given by the Hausdorff distance
- Example: the sets $\{x, y\}$ and $\{u, v, w\}$ are close in Sub(V, t)

Lemma

$I_t \to \mathbb{Z}_{\geq 0}, S \mapsto |S|$ is continuous

- The sets $\{x, y\}$ and $\{u, v, w\}$ are in different connected components in I_t

Positive kernels

- A function $f \in \mathcal{C}(V_t \times V_t)_{\mathsf{sym}}$ is a *positive kernel* if

$$(f(x_i, x_j))_{i,j=1}^{m}$$

  is positive semidefinite for all m and $x_1, \ldots, x_m \in V_t$
- Cone of positive (definite) kernels: $\mathcal{C}(V_t \times V_t)_{\succeq 0}$

Measures of positive type

- $M(V_t \times V_t)_{\succeq 0}$ is the cone dual to $\mathcal{C}(V_t \times V_t)_{\succeq 0}$
- The elements in $M(V_t \times V_t)_{\succeq 0}$ are called *positive definite measures*
- Define the operator $A_t \colon \mathcal{C}(V_t \times V_t)_{\text{sym}} \to \mathcal{C}(I_{2t})$ by

$$A_t f(S) = \sum_{J, J' \in V_t \colon J \cup J' = S} f(J, J')$$

- The *measures of positive type* on I_{2t}:

$$\left\{ \lambda \in M(I_{2t}) \colon A_t^* \lambda \in M(V_t \times V_t)_{\succeq 0} \right\}$$

- A measure λ on a locally compact group Γ is of positive type if it defines a positive linear functional on the group algebra:

$$\lambda(f^* * f) \geq 0 \text{ for all } f \in \mathcal{C}(\Gamma)$$

- For $f, g \in \mathcal{C}(V_t)$, let $f^* = f$ and $f * g = A_t(f \otimes g)$

The hierarchie

- Generalization of the Lasserre hierarchy to infinite graphs:

$$\vartheta_t = \inf \Big\{ f(\emptyset, \emptyset) \colon f \in \mathcal{C}(V_t \times V_t)_{\succeq 0}, \\ \mathbb{1}_{I_1} + A_t f \in \mathcal{C}(I_{2t})_{\leq 0} \Big\}$$

- Conic duality gives the dual chain

$$\vartheta_t^* = \sup \Big\{ \lambda(I_1) \colon \lambda \in M(I_{2t})_{\geq 0}, \\ \delta_\emptyset \otimes \delta_\emptyset + A_t^* \lambda \in M(V_t \times V_t)_{\succeq 0} \Big\}$$

Theorem

1. $\vartheta_t = \vartheta_t^*$ for all t
2. $\alpha \leq \ldots \leq \vartheta_3^* \leq \vartheta_2^* \leq \vartheta_1^*$
3. $\vartheta_\alpha^* = \alpha$

Strong duality

- To prove strong duality we use a closed cone condition
- We need to show that the cone

$$K = \{(A_t^*\lambda - \mu, \lambda(I_1)) \colon \mu \in M(V_t \times V_t)_{\succeq 0}, \lambda \in M(I_{2t})_{\geq 0}\}$$

  is closed in $M(V_t \times V_t)_{\mathrm{sym}} \times \mathbb{R}$
- Idea: $K = K_1 - K_2$ (Minkowski difference)

Lemma (Klee 1955)

If K_1 and K_2 are closed convex cones in a topological vector space, K_1 is locally compact, and $K_1 \cap K_2 = \{0\}$, then $K_1 - K_2$ is closed.

Finite convergence

- We write the αth step of the hierarchy as

$$\Theta = \max\{\lambda(I_1) \colon \lambda \in M(I), \lambda(\{\emptyset\}) = 1, A^*\lambda \succeq 0\}$$

  where I is the collection of all independent sets
- Claim: $\Theta = \alpha$
- Given an independent set S, $\chi_S = \sum_{J \subseteq S} \delta_J$ is feasible for Θ
- λ feasible $\Rightarrow \lambda = \int \chi_S d\sigma(S)$
- We show that σ is a probability measure
- $\Theta = \max\{\int |S| d\sigma(S) \colon \sigma \in \mathcal{P}(I)\}$

Thank you!

D. de Laat, F. Vallentin, *A semidefinite programming hierarchy for geometric packing problems*, in preparation.