Using noncommutative polynomial optimization for matrix factorization ranks

Sander Gribling (CWI/QuSoft)
David de Laat (CWI/QuSoft)
Monique Laurent (CWI/Tilburg/QuSoft)

SIAM Conference on Optimization, 25 May 2017, Vancouver

Symmetric matrix factorization ranks

Symmetric matrix factorization ranks

PSD matrices

$A \in \mathbb{R}^{n \times n}$ is PSD if there are $a_1, \ldots, a_n \in \mathbb{R}^d$ with $A_{ij} = a_i^\mathsf{T} a_j$

Symmetric matrix factorization ranks

PSD matrices

$A \in \mathbb{R}^{n \times n}$ is PSD if there are $a_1, \ldots, a_n \in \mathbb{R}^d$ with $A_{ij} = a_i^\mathsf{T} a_j$

$\mathrm{rank}(A) =$ smallest possible d;

Symmetric matrix factorization ranks

PSD matrices

$A \in \mathbb{R}^{n \times n}$ is PSD if there are $a_1, \ldots, a_n \in \mathbb{R}^d$ with $A_{ij} = a_i^\mathsf{T} a_j$

$\mathrm{rank}(A) =$ smallest possible d; Easy to compute;

Symmetric matrix factorization ranks

PSD matrices

$A \in \mathbb{R}^{n\times n}$ is PSD if there are $a_1, \ldots, a_n \in \mathbb{R}^d$ with $A_{ij} = a_i^\mathsf{T} a_j$

$\mathrm{rank}(A) =$ smallest possible d; Easy to compute; $d \leq n$

Symmetric matrix factorization ranks

PSD matrices

$A \in \mathbb{R}^{n\times n}$ is PSD if there are $a_1, \ldots, a_n \in \mathbb{R}^d$ with $A_{ij} = a_i^\mathsf{T} a_j$

$\mathrm{rank}(A) =$ smallest possible d; Easy to compute; $d \leq n$

CP matrices

$A \in \mathbb{R}^{n\times n}$ is CP if there are $a_1, \ldots, a_n \in \mathbb{R}^d_+$ with $A_{ij} = a_i^\mathsf{T} a_j$

Symmetric matrix factorization ranks

PSD matrices

$A \in \mathbb{R}^{n\times n}$ is PSD if there are $a_1, \dots, a_n \in \mathbb{R}^d$ with $A_{ij} = a_i^\mathsf{T} a_j$

$\mathrm{rank}(A) =$ smallest possible d; Easy to compute; $d \leq n$

CP matrices

$A \in \mathbb{R}^{n\times n}$ is CP if there are $a_1, \dots, a_n \in \mathbb{R}_+^d$ with $A_{ij} = a_i^\mathsf{T} a_j$

$\text{cp-rank}(A) =$ smallest possible d;

Symmetric matrix factorization ranks

PSD matrices

$A \in \mathbb{R}^{n\times n}$ is PSD if there are $a_1, \ldots, a_n \in \mathbb{R}^d$ with $A_{ij} = a_i^\mathsf{T} a_j$

$\mathrm{rank}(A) =$ smallest possible d; Easy to compute; $d \leq n$

CP matrices

$A \in \mathbb{R}^{n\times n}$ is CP if there are $a_1, \ldots, a_n \in \mathbb{R}^d_+$ with $A_{ij} = a_i^\mathsf{T} a_j$

$\mathrm{cp\text{-}rank}(A) =$ smallest possible d; Hard to compute;

Symmetric matrix factorization ranks

PSD matrices

$A \in \mathbb{R}^{n\times n}$ is PSD if there are $a_1, \ldots, a_n \in \mathbb{R}^d$ with $A_{ij} = a_i^\mathsf{T} a_j$

$\mathrm{rank}(A) =$ smallest possible d; Easy to compute; $d \leq n$

CP matrices

$A \in \mathbb{R}^{n\times n}$ is CP if there are $a_1, \ldots, a_n \in \mathbb{R}^d_+$ with $A_{ij} = a_i^\mathsf{T} a_j$

$\mathrm{cp\text{-}rank}(A) =$ smallest possible d; Hard to compute;

If A is CP, then $d \leq \binom{n+1}{2} + 1$

Symmetric matrix factorization ranks

PSD matrices

$A \in \mathbb{R}^{n\times n}$ is PSD if there are $a_1, \dots, a_n \in \mathbb{R}^d$ with $A_{ij} = a_i^\mathsf{T} a_j$

$\mathrm{rank}(A) =$ smallest possible d; Easy to compute; $d \le n$

CP matrices

$A \in \mathbb{R}^{n\times n}$ is CP if there are $a_1, \dots, a_n \in \mathbb{R}^d_+$ with $A_{ij} = a_i^\mathsf{T} a_j$

$\mathrm{cp\text{-}rank}(A) =$ smallest possible d; Hard to compute;

If A is CP, then $d \le \binom{n+1}{2} + 1$

CPSD matrices

$A \in \mathbb{R}^{n\times n}$ is CPSD if there are are Hermitian PSD matrices

$X_1, \dots, X_n \in \mathbb{C}^{d\times d}$ with $A_{ij} = \mathrm{Tr}(X_i X_j)$

Symmetric matrix factorization ranks

PSD matrices

$A \in \mathbb{R}^{n\times n}$ is PSD if there are $a_1, \ldots, a_n \in \mathbb{R}^d$ with $A_{ij} = a_i^\mathsf{T} a_j$

$\mathrm{rank}(A)$ = smallest possible d; Easy to compute; $d \leq n$

CP matrices

$A \in \mathbb{R}^{n\times n}$ is CP if there are $a_1, \ldots, a_n \in \mathbb{R}^d_+$ with $A_{ij} = a_i^\mathsf{T} a_j$

$\mathrm{cp\text{-}rank}(A)$ = smallest possible d; Hard to compute;

If A is CP, then $d \leq \binom{n+1}{2} + 1$

CPSD matrices

$A \in \mathbb{R}^{n\times n}$ is CPSD if there are are Hermitian PSD matrices

$X_1, \ldots, X_n \in \mathbb{C}^{d\times d}$ with $A_{ij} = \mathrm{Tr}(X_i X_j)$

$\mathrm{cpsd\text{-}rank}(A)$ = smallest possible d;

Symmetric matrix factorization ranks

PSD matrices

$A \in \mathbb{R}^{n\times n}$ is PSD if there are $a_1,\ldots,a_n \in \mathbb{R}^d$ with $A_{ij} = a_i^\mathsf{T} a_j$

$\mathrm{rank}(A) =$ smallest possible d; Easy to compute; $d \le n$

CP matrices

$A \in \mathbb{R}^{n\times n}$ is CP if there are $a_1,\ldots,a_n \in \mathbb{R}^d_+$ with $A_{ij} = a_i^\mathsf{T} a_j$

$\mathrm{cp\text{-}rank}(A) =$ smallest possible d; Hard to compute;

If A is CP, then $d \le \binom{n+1}{2} + 1$

CPSD matrices

$A \in \mathbb{R}^{n\times n}$ is CPSD if there are are Hermitian PSD matrices

$X_1,\ldots,X_n \in \mathbb{C}^{d\times d}$ with $A_{ij} = \mathrm{Tr}(X_i X_j)$

$\mathrm{cpsd\text{-}rank}(A) =$ smallest possible d; Hard to compute;

Symmetric matrix factorization ranks

PSD matrices

$A \in \mathbb{R}^{n\times n}$ is PSD if there are $a_1, \ldots, a_n \in \mathbb{R}^d$ with $A_{ij} = a_i^\mathsf{T} a_j$

$\mathrm{rank}(A) =$ smallest possible d; Easy to compute; $d \leq n$

CP matrices

$A \in \mathbb{R}^{n\times n}$ is CP if there are $a_1, \ldots, a_n \in \mathbb{R}_+^d$ with $A_{ij} = a_i^\mathsf{T} a_j$

$\mathrm{cp\text{-}rank}(A) =$ smallest possible d; Hard to compute;

If A is CP, then $d \leq \binom{n+1}{2} + 1$

CPSD matrices

$A \in \mathbb{R}^{n\times n}$ is CPSD if there are are Hermitian PSD matrices

$X_1, \ldots, X_n \in \mathbb{C}^{d\times d}$ with $A_{ij} = \mathrm{Tr}(X_i X_j)$

$\mathrm{cpsd\text{-}rank}(A) =$ smallest possible d; Hard to compute;

There is no upper bound on d depending only on n [Slofstra, 2017]

Symmetric matrix factorization ranks

PSD matrices

$A \in \mathbb{R}^{n \times n}$ is PSD if there are $a_1, \ldots, a_n \in \mathbb{R}^d$ with $A_{ij} = a_i^\mathsf{T} a_j$

$\mathrm{rank}(A) =$ smallest possible d; Easy to compute; $d \leq n$

CP matrices

$A \in \mathbb{R}^{n \times n}$ is CP if there are $a_1, \ldots, a_n \in \mathbb{R}^d_+$ with $A_{ij} = a_i^\mathsf{T} a_j$

$\mathrm{cp\text{-}rank}(A) =$ smallest possible d; Hard to compute;

If A is CP, then $d \leq \binom{n+1}{2} + 1$

CPSD matrices

$A \in \mathbb{R}^{n \times n}$ is CPSD if there are are Hermitian PSD matrices

$X_1, \ldots, X_n \in \mathbb{C}^{d \times d}$ with $A_{ij} = \mathrm{Tr}(X_i X_j)$

$\mathrm{cpsd\text{-}rank}(A) =$ smallest possible d; Hard to compute;

There is no upper bound on d depending only on n [Slofstra, 2017]

Symmetric matrix factorization ranks

PSD matrices

$A \in \mathbb{R}^{n\times n}$ is PSD if there are $a_1, \ldots, a_n \in \mathbb{R}^d$ with $A_{ij} = a_i^\mathsf{T} a_j$

$\mathrm{rank}(A) =$ smallest possible d; Easy to compute; $d \leq n$

CP matrices

$A \in \mathbb{R}^{n\times n}$ is CP if there are $a_1, \ldots, a_n \in \mathbb{R}^d_+$ with $A_{ij} = a_i^\mathsf{T} a_j$

$\mathrm{cp\text{-}rank}(A) =$ smallest possible d; Hard to compute;

If A is CP, then $d \leq \binom{n+1}{2} + 1$

CPSD matrices

$A \in \mathbb{R}^{n\times n}$ is CPSD if there are are Hermitian PSD matrices

$X_1, \ldots, X_n \in \mathbb{C}^{d\times d}$ with $A_{ij} = \mathrm{Tr}(X_i X_j)$

$\mathrm{cpsd\text{-}rank}(A) =$ smallest possible d; Hard to compute;

There is no upper bound on d depending only on n [Slofstra, 2017]

CP matrices $\subseteq$ CPSD matrices $\subseteq$ PSD matrices

Symmetric matrix factorization ranks

PSD matrices

$A \in \mathbb{R}^{n\times n}$ is PSD if there are $a_1,\ldots,a_n \in \mathbb{R}^d$ with $A_{ij} = a_i^\mathsf{T} a_j$

$\mathrm{rank}(A) =$ smallest possible d; Easy to compute; $d \leq n$

CP matrices

$A \in \mathbb{R}^{n\times n}$ is CP if there are $a_1,\ldots,a_n \in \mathbb{R}^d_+$ with $A_{ij} = a_i^\mathsf{T} a_j$

$\mathrm{cp\text{-}rank}(A) =$ smallest possible d; Hard to compute;

If A is CP, then $d \leq \binom{n+1}{2} + 1$

CPSD matrices

$A \in \mathbb{R}^{n\times n}$ is CPSD if there are are Hermitian PSD matrices

$X_1,\ldots,X_n \in \mathbb{C}^{d\times d}$ with $A_{ij} = \mathrm{Tr}(X_i X_j)$

$\mathrm{cpsd\text{-}rank}(A) =$ smallest possible d; Hard to compute;

There is no upper bound on d depending only on n [Slofstra, 2017]

CP matrices $\subseteq$ CPSD matrices $\subseteq$ PSD matrices

Goal: Find lower bounds for matrix factorization ranks

Connection to quantum information theory

- CPSD cone was studied by Piovesan and Laurent in relation to quantum graph parameters

Connection to quantum information theory

- CPSD cone was studied by Piovesan and Laurent in relation to quantum graph parameters
- Connections to entanglement dimensions of bipartite quantum correlations $p(a, b|s, t)$ [Sikora–Varvitsiotis 2015], [Mančinska–Roberson 2014]

Connection to quantum information theory

- CPSD cone was studied by Piovesan and Laurent in relation to quantum graph parameters
- Connections to entanglement dimensions of bipartite quantum correlations $p(a, b|s, t)$ [Sikora–Varvitsiotis 2015], [Mančinska–Roberson 2014]
- Corresponding matrix $(A_p)_{(s,a),(t,b)} = p(a, b|s, t)$

Connection to quantum information theory

- CPSD cone was studied by Piovesan and Laurent in relation to quantum graph parameters
- Connections to entanglement dimensions of bipartite quantum correlations $p(a, b|s, t)$ [Sikora–Varvitsiotis 2015], [Mančinska–Roberson 2014]
- Corresponding matrix $(A_p)_{(s,a),(t,b)} = p(a, b|s, t)$
- If p is a "synchronous quantum correlation", then A_p is CPSD

Connection to quantum information theory

- CPSD cone was studied by Piovesan and Laurent in relation to quantum graph parameters
- Connections to entanglement dimensions of bipartite quantum correlations $p(a, b|s, t)$ [Sikora–Varvitsiotis 2015], [Mančinska–Roberson 2014]
- Corresponding matrix $(A_p)_{(s,a),(t,b)} = p(a, b|s, t)$
- If p is a "synchronous quantum correlation", then A_p is CPSD
- The smallest dimension to realize it is $\mathrm{cpsd\text{-}rank}(A_p)$

Connection to quantum information theory

- CPSD cone was studied by Piovesan and Laurent in relation to quantum graph parameters
- Connections to entanglement dimensions of bipartite quantum correlations $p(a, b|s, t)$ [Sikora–Varvitsiotis 2015], [Mančinska–Roberson 2014]
- Corresponding matrix $(A_p)_{(s,a),(t,b)} = p(a, b|s, t)$
- If p is a "synchronous quantum correlation", then A_p is CPSD
- The smallest dimension to realize it is $\mathrm{cpsd\text{-}rank}(A_p)$
- Combine proofs from above refs and [Paulsen–Severini–Stahlke–Todorov–Winter 2016]

Polynomial optimization

Commutative polynomial optimization (Lasserre, Parrilo, ...):

Polynomial optimization

Commutative polynomial optimization (Lasserre, Parrilo, ...):

- Let $S \cup \{f\} \subseteq \mathbb{R}[x_1, \ldots, x_n]$

Polynomial optimization

Commutative polynomial optimization (Lasserre, Parrilo, ...):

- Let $S \cup \{f\} \subseteq \mathbb{R}[x_1, \ldots, x_n]$
- $\inf \big\{ f(x) : x \in \mathbb{R}^n,\ g(x) \geq 0 \text{ for } g \in S \big\}$

Polynomial optimization

Commutative polynomial optimization (Lasserre, Parrilo, ...):

- Let $S \cup \{f\} \subseteq \mathbb{R}[x_1, \ldots, x_n]$
- $\inf \Big\{ f(x) : x \in \mathbb{R}^n,\ g(x) \geq 0 \text{ for } g \in S \Big\}$
- Hierarchy of semidefinite programming lower bounds based on moments (primal) and sums of squares (dual)

Polynomial optimization

Commutative polynomial optimization (Lasserre, Parrilo, ...):

- Let $S \cup \{f\} \subseteq \mathbb{R}[x_1, \dots, x_n]$
- $\inf \left\{ f(x) : x \in \mathbb{R}^n,\, g(x) \geq 0 \text{ for } g \in S \right\}$
- Hierarchy of semidefinite programming lower bounds based on moments (primal) and sums of squares (dual)
- Asymptotic convergence under technical condition

Polynomial optimization

Commutative polynomial optimization (Lasserre, Parrilo, ...):

- Let $S \cup \{f\} \subseteq \mathbb{R}[x_1, \ldots, x_n]$
- inf $\big\{ f(x) : x \in \mathbb{R}^n,\ g(x) \geq 0 \text{ for } g \in S \big\}$
- Hierarchy of semidefinite programming lower bounds based on moments (primal) and sums of squares (dual)
- Asymptotic convergence under technical condition

Eigenvalue optimization (Acín, Navascues, Pironio, ...) and tracial optimization (Burgdorf, Cafuta, Klep, Povh, Schweighofer, ...):

Polynomial optimization

Commutative polynomial optimization (Lasserre, Parrilo, ...):

- Let $S \cup \{f\} \subseteq \mathbb{R}[x_1, \ldots, x_n]$
- $\inf \big\{ f(x) : x \in \mathbb{R}^n,\ g(x) \geq 0 \text{ for } g \in S \big\}$
- Hierarchy of semidefinite programming lower bounds based on moments (primal) and sums of squares (dual)
- Asymptotic convergence under technical condition

Eigenvalue optimization (Acín, Navascues, Pironio, ...) and tracial optimization (Burgdorf, Cafuta, Klep, Povh, Schweighofer, ...):

- Let $S \cup \{f\} \subseteq \mathbb{R}\langle x_1, \ldots, x_n \rangle$

Polynomial optimization

Commutative polynomial optimization (Lasserre, Parrilo, ...):

- Let $S \cup \{f\} \subseteq \mathbb{R}[x_1, \ldots, x_n]$
- $\inf \left\{ f(x) : x \in \mathbb{R}^n, \, g(x) \geq 0 \text{ for } g \in S \right\}$
- Hierarchy of semidefinite programming lower bounds based on moments (primal) and sums of squares (dual)
- Asymptotic convergence under technical condition

Eigenvalue optimization (Acín, Navascues, Pironio, ...) and tracial optimization (Burgdorf, Cafuta, Klep, Povh, Schweighofer, ...):

- Let $S \cup \{f\} \subseteq \mathbb{R}\langle x_1, \ldots, x_n \rangle$
- We can evaluate a noncommutative polynomial at a tuple $\mathbf{X} = (X_1, \ldots, X_n)$ of matrices

Polynomial optimization

Commutative polynomial optimization (Lasserre, Parrilo, ...):

- Let $S \cup \{f\} \subseteq \mathbb{R}[x_1, \ldots, x_n]$
- $\inf \big\{ f(x) : x \in \mathbb{R}^n,\, g(x) \geq 0 \text{ for } g \in S \big\}$
- Hierarchy of semidefinite programming lower bounds based on moments (primal) and sums of squares (dual)
- Asymptotic convergence under technical condition

Eigenvalue optimization (Acín, Navascues, Pironio, ...) and tracial optimization (Burgdorf, Cafuta, Klep, Povh, Schweighofer, ...):

- Let $S \cup \{f\} \subseteq \mathbb{R}\langle x_1, \ldots, x_n \rangle$
- We can evaluate a noncommutative polynomial at a tuple $\mathbf{X} = (X_1, \ldots, X_n)$ of matrices
- $\inf\{\mathrm{tr}(f(\mathbf{X})) : d \in \mathbb{N},\, X_1, \ldots, X_n \in H^d,\, g(\mathbf{X}) \succeq 0 \text{ for } g \in S\}$

Polynomial optimization

Commutative polynomial optimization (Lasserre, Parrilo, ...):

- Let $S \cup \{f\} \subseteq \mathbb{R}[x_1, \ldots, x_n]$
- $\inf \big\{ f(x) : x \in \mathbb{R}^n,\, g(x) \geq 0 \text{ for } g \in S \big\}$
- Hierarchy of semidefinite programming lower bounds based on moments (primal) and sums of squares (dual)
- Asymptotic convergence under technical condition

Eigenvalue optimization (Acín, Navascues, Pironio, ...) and tracial optimization (Burgdorf, Cafuta, Klep, Povh, Schweighofer, ...):

- Let $S \cup \{f\} \subseteq \mathbb{R}\langle x_1, \ldots, x_n \rangle$
- We can evaluate a noncommutative polynomial at a tuple $\mathbf{X} = (X_1, \ldots, X_n)$ of matrices
- $\inf\{\mathrm{tr}(f(\mathbf{X})) : d \in \mathbb{N},\, X_1, \ldots, X_n \in H^d,\, g(\mathbf{X}) \succeq 0 \text{ for } g \in S\}$

Commutative polynomial optimization is used by Nie for testing membership in the CP cone and computing tensor nuclear norms

Lower bounding the cpsd-rank using tracial optimization

Let $A \in \mathbb{R}^{n \times n}$ be a CPSD matrix and set $d = \mathrm{cpsd\text{-}rank}(A)$

Lower bounding the cpsd-rank using tracial optimization

Let $A \in \mathbb{R}^{n\times n}$ be a CPSD matrix and set $d = \mathrm{cpsd\text{-}rank}(A)$

$X_1, \ldots, X_n \in \mathbb{C}^{d\times d}$ Hermitian PSD matrices with $A_{ij} = \mathrm{Tr}(X_i X_j)$

Lower bounding the cpsd-rank using tracial optimization

Let $A \in \mathbb{R}^{n\times n}$ be a CPSD matrix and set $d = \mathrm{cpsd\text{-}rank}(A)$

$X_1, \ldots, X_n \in \mathbb{C}^{d\times d}$ Hermitian PSD matrices with $A_{ij} = \mathrm{Tr}(X_i X_j)$

$\mathbb{R}\langle x_1, \ldots, x_n\rangle$: $*$-algebra of noncommutative polynomials in n vars

Lower bounding the cpsd-rank using tracial optimization

Let $A \in \mathbb{R}^{n\times n}$ be a CPSD matrix and set $d = \mathrm{cpsd\text{-}rank}(A)$

$X_1, \ldots, X_n \in \mathbb{C}^{d\times d}$ Hermitian PSD matrices with $A_{ij} = \mathrm{Tr}(X_i X_j)$

$\mathbb{R}\langle x_1, \ldots, x_n\rangle$: $*$-algebra of noncommutative polynomials in n vars

Define a linear form $L_X \in \mathbb{R}\langle x_1, \ldots, x_n\rangle^*$ by

$$L_X(p) = \mathrm{Re}(\mathrm{Tr}(p(X_1, \ldots, X_n)))$$

Lower bounding the cpsd-rank using tracial optimization

Let $A \in \mathbb{R}^{n \times n}$ be a CPSD matrix and set $d = \text{cpsd-rank}(A)$

$X_1, \ldots, X_n \in \mathbb{C}^{d \times d}$ Hermitian PSD matrices with $A_{ij} = \mathrm{Tr}(X_i X_j)$

$\mathbb{R}\langle x_1, \ldots, x_n \rangle$: $*$-algebra of noncommutative polynomials in n vars

Define a linear form $L_X \in \mathbb{R}\langle x_1, \ldots, x_n \rangle^*$ by

$$L_X(p) = \mathrm{Re}(\mathrm{Tr}(p(X_1, \ldots, X_n)))$$

We have $L_X(1) = \mathrm{Re}(\mathrm{Tr}(I_d)) = d = \text{cpsd-rank}(A)$

Lower bounding the cpsd-rank using tracial optimization

Let $A \in \mathbb{R}^{n \times n}$ be a CPSD matrix and set $d = \mathrm{cpsd\text{-}rank}(A)$

$X_1, \ldots, X_n \in \mathbb{C}^{d \times d}$ Hermitian PSD matrices with $A_{ij} = \mathrm{Tr}(X_i X_j)$

$\mathbb{R}\langle x_1, \ldots, x_n \rangle$: $*$-algebra of noncommutative polynomials in n vars

Define a linear form $L_X \in \mathbb{R}\langle x_1, \ldots, x_n \rangle^*$ by

$$L_X(p) = \mathrm{Re}(\mathrm{Tr}(p(X_1, \ldots, X_n)))$$

We have $L_X(1) = \mathrm{Re}(\mathrm{Tr}(I_d)) = d = \mathrm{cpsd\text{-}rank}(A)$

We obtain a relaxation by minimizing $L(1)$ over all linear forms L that satisfy some computationally tractable properties of L_X

Lower bounding the cpsd-rank using tracial optimization

Let $A \in \mathbb{R}^{n \times n}$ be a CPSD matrix and set $d = \mathrm{cpsd\text{-}rank}(A)$

$X_1, \ldots, X_n \in \mathbb{C}^{d \times d}$ Hermitian PSD matrices with $A_{ij} = \mathrm{Tr}(X_i X_j)$

$\mathbb{R}\langle x_1, \ldots, x_n \rangle$: $*$-algebra of noncommutative polynomials in n vars

Define a linear form $L_X \in \mathbb{R}\langle x_1, \ldots, x_n \rangle^*$ by

$$L_X(p) = \mathrm{Re}(\mathrm{Tr}(p(X_1, \ldots, X_n)))$$

We have $L_X(1) = \mathrm{Re}(\mathrm{Tr}(I_d)) = d = \mathrm{cpsd\text{-}rank}(A)$

We obtain a relaxation by minimizing $L(1)$ over all linear forms L that satisfy some computationally tractable properties of L_X

Symmetric and tracial: $L_X(p^*) = L_X(p)$ and $L_X(pq) = L_X(qp)$

Lower bounding the cpsd-rank using tracial optimization

Let $A \in \mathbb{R}^{n \times n}$ be a CPSD matrix and set $d = \text{cpsd-rank}(A)$

$X_1, \ldots, X_n \in \mathbb{C}^{d \times d}$ Hermitian PSD matrices with $A_{ij} = \mathrm{Tr}(X_i X_j)$

$\mathbb{R}\langle x_1, \ldots, x_n \rangle$: $*$-algebra of noncommutative polynomials in n vars

Define a linear form $L_X \in \mathbb{R}\langle x_1, \ldots, x_n \rangle^*$ by

$$L_X(p) = \mathrm{Re}(\mathrm{Tr}(p(X_1, \ldots, X_n)))$$

We have $L_X(1) = \mathrm{Re}(\mathrm{Tr}(I_d)) = d = \text{cpsd-rank}(A)$

We obtain a relaxation by minimizing $L(1)$ over all linear forms L that satisfy some computationally tractable properties of L_X

Symmetric and tracial: $L_X(p^*) = L_X(p)$ and $L_X(pq) = L_X(qp)$
Positive: $L_X(p^* p) \geq 0$

Lower bounding the cpsd-rank using tracial optimization

Let $A \in \mathbb{R}^{n\times n}$ be a CPSD matrix and set $d = \mathrm{cpsd\text{-}rank}(A)$

$X_1, \ldots, X_n \in \mathbb{C}^{d\times d}$ Hermitian PSD matrices with $A_{ij} = \mathrm{Tr}(X_i X_j)$

$\mathbb{R}\langle x_1, \ldots, x_n\rangle$: $*$-algebra of noncommutative polynomials in n vars

Define a linear form $L_X \in \mathbb{R}\langle x_1, \ldots, x_n\rangle^*$ by

$$L_X(p) = \mathrm{Re}(\mathrm{Tr}(p(X_1, \ldots, X_n)))$$

We have $L_X(1) = \mathrm{Re}(\mathrm{Tr}(I_d)) = d = \mathrm{cpsd\text{-}rank}(A)$

We obtain a relaxation by minimizing $L(1)$ over all linear forms L that satisfy some computationally tractable properties of L_X

Symmetric and tracial: $L_X(p^*) = L_X(p)$ and $L_X(pq) = L_X(qp)$
Positive: $L_X(p^*p) \geq 0$
Linear conditions: $L_X(x_i x_j) = A_{ij}$

Lower bounding the cpsd-rank using tracial optimization

Let $A \in \mathbb{R}^{n \times n}$ be a CPSD matrix and set $d = \mathrm{cpsd\text{-}rank}(A)$

$X_1, \ldots, X_n \in \mathbb{C}^{d \times d}$ Hermitian PSD matrices with $A_{ij} = \mathrm{Tr}(X_i X_j)$

$\mathbb{R}\langle x_1, \ldots, x_n \rangle$: $*$-algebra of noncommutative polynomials in n vars

Define a linear form $L_X \in \mathbb{R}\langle x_1, \ldots, x_n \rangle^*$ by

$$L_X(p) = \mathrm{Re}(\mathrm{Tr}(p(X_1, \ldots, X_n)))$$

We have $L_X(1) = \mathrm{Re}(\mathrm{Tr}(I_d)) = d = \mathrm{cpsd\text{-}rank}(A)$

We obtain a relaxation by minimizing $L(1)$ over all linear forms L that satisfy some computationally tractable properties of L_X

Symmetric and tracial: $L_X(p^*) = L_X(p)$ and $L_X(pq) = L_X(qp)$

Positive: $L_X(p^* p) \geq 0$

Linear conditions: $L_X(x_i x_j) = A_{ij}$

Localizing conditions: $L_X(p^*(\sqrt{A_{ii}} x_i - x_i^2) p) \geq 0$

Truncate to obtain a semidefinite programming hierarchy

$\mathbb{R}\langle x_1, \ldots, x_n \rangle_{2t}$ noncommututative polynomials with $\deg \leq 2t$

Truncate to obtain a semidefinite programming hierarchy

$\mathbb{R}\langle x_1, \ldots, x_n \rangle_{2t}$ noncommututative polynomials with $\deg \leq 2t$

Let $S \subseteq \mathbb{R}\langle \mathbf{x} \rangle = \mathbb{R}\langle x_1, \ldots, x_n \rangle$

Truncate to obtain a semidefinite programming hierarchy

$\mathbb{R}\langle x_1, \ldots, x_n \rangle_{2t}$ noncommututative polynomials with $\deg \leq 2t$

Let $S \subseteq \mathbb{R}\langle \mathbf{x} \rangle = \mathbb{R}\langle x_1, \ldots, x_n \rangle$

Quadratic module: $\mathcal{M}(S) = \mathrm{cone}\{p^* g p : g \in S \cup \{1\},\, p \in \mathbb{R}\langle \mathbf{x} \rangle\}$

Truncate to obtain a semidefinite programming hierarchy

$\mathbb{R}\langle x_1, \ldots, x_n \rangle_{2t}$ noncommutututative polynomials with $\deg \leq 2t$

Let $S \subseteq \mathbb{R}\langle \mathbf{x} \rangle = \mathbb{R}\langle x_1, \ldots, x_n \rangle$

Quadratic module: $\mathcal{M}(S) = \mathrm{cone}\{p^* g p : g \in S \cup \{1\},\, p \in \mathbb{R}\langle \mathbf{x} \rangle\}$

Truncated quadratic module:
$\mathcal{M}_{2t}(S) = \mathrm{cone}\{p^* g p : g \in S \cup \{1\},\, p \in \mathbb{R}\langle \mathbf{x} \rangle,\, \mathrm{deg}(p^* g p) \leq 2t\}$

Truncate to obtain a semidefinite programming hierarchy

$\mathbb{R}\langle x_1, \ldots, x_n \rangle_{2t}$ noncommutututative polynomials with $\deg \leq 2t$

Let $S \subseteq \mathbb{R}\langle \mathbf{x} \rangle = \mathbb{R}\langle x_1, \ldots, x_n \rangle$

Quadratic module: $\mathcal{M}(S) = \mathrm{cone}\{p^* g p : g \in S \cup \{1\},\, p \in \mathbb{R}\langle \mathbf{x} \rangle\}$

Truncated quadratic module:
$\mathcal{M}_{2t}(S) = \mathrm{cone}\{p^* g p : g \in S \cup \{1\},\, p \in \mathbb{R}\langle \mathbf{x} \rangle,\, \deg(p^* g p) \leq 2t\}$

$$\xi_t^{\mathrm{cpsd}}(A) = \min\Big\{L(1) : L \in \mathbb{R}\langle x_1, \ldots, x_n \rangle_{2t}^* \text{ tracial and symmetric}, \\ (L(x_i x_j)) = A, \\ L \geq 0 \quad \text{on} \quad \mathcal{M}_{2t}\big(\{\sqrt{A_{ii}}x_i - x_i^2 : i \in [n]\}\big)\Big\}$$

Truncate to obtain a semidefinite programming hierarchy

$\mathbb{R}\langle x_1,\ldots,x_n\rangle_{2t}$ noncommutututative polynomials with $\deg \leq 2t$

Let $S \subseteq \mathbb{R}\langle \mathbf{x}\rangle = \mathbb{R}\langle x_1,\ldots,x_n\rangle$

Quadratic module: $\mathcal{M}(S) = \mathrm{cone}\{p^*gp : g \in S \cup \{1\},\ p \in \mathbb{R}\langle \mathbf{x}\rangle\}$

Truncated quadratic module:
$\mathcal{M}_{2t}(S) = \mathrm{cone}\{p^*gp : g \in S \cup \{1\},\ p \in \mathbb{R}\langle \mathbf{x}\rangle,\ \deg(p^*gp) \leq 2t\}$

$$\xi_t^{\mathrm{cpsd}}(A) = \min\Big\{L(1) : L \in \mathbb{R}\langle x_1,\ldots,x_n\rangle_{2t}^* \text{ tracial and symmetric}, \\ (L(x_ix_j)) = A, \\ L \geq 0 \quad \text{on} \quad \mathcal{M}_{2t}\big(\{\sqrt{A_{ii}}x_i - x_i^2 : i \in [n]\}\big)\Big\}$$

$$\xi_1^{\mathrm{cpsd}}(A) \leq \ldots \leq \xi_\infty^{\mathrm{cpsd}}(A) \leq \xi_*^{\mathrm{cpsd}}(A) \leq \mathrm{cpsd\text{-}rank}(A)$$

Truncate to obtain a semidefinite programming hierarchy

$\mathbb{R}\langle x_1, \ldots, x_n \rangle_{2t}$ noncommututative polynomials with $\deg \leq 2t$

Let $S \subseteq \mathbb{R}\langle \mathbf{x} \rangle = \mathbb{R}\langle x_1, \ldots, x_n \rangle$

Quadratic module: $\mathcal{M}(S) = \mathrm{cone}\{p^* g p : g \in S \cup \{1\},\, p \in \mathbb{R}\langle \mathbf{x} \rangle\}$

Truncated quadratic module:
$\mathcal{M}_{2t}(S) = \mathrm{cone}\{p^* g p : g \in S \cup \{1\},\, p \in \mathbb{R}\langle \mathbf{x} \rangle,\, \deg(p^* g p) \leq 2t\}$

$$\xi_t^{\mathrm{cpsd}}(A) = \min\Big\{ L(1) : L \in \mathbb{R}\langle x_1, \ldots, x_n \rangle_{2t}^* \text{ tracial and symmetric}, \\ (L(x_i x_j)) = A, \\ L \geq 0 \quad \text{on} \quad \mathcal{M}_{2t}\big(\{\sqrt{A_{ii}}\, x_i - x_i^2 : i \in [n]\}\big) \Big\}$$

$$\xi_1^{\mathrm{cpsd}}(A) \leq \ldots \leq \xi_\infty^{\mathrm{cpsd}}(A) \leq \xi_*^{\mathrm{cpsd}}(A) \leq \mathrm{cpsd\text{-}rank}(A)$$

$\xi_*^{\mathrm{cpsd}}(A)$ is $\xi_\infty^{\mathrm{cpsd}}(A)$ with the extra constraint $\mathrm{rank}(M(L)) < \infty$

$\xi_{\infty}^{\mathrm{cpsd}}(A)$ and $\xi_{*}^{\mathrm{cpsd}}(A)$

- We have $\xi_t^{\mathrm{cpsd}}(A) \to \xi_{\infty}^{\mathrm{cpsd}}(A)$, and if $\xi_t^{\mathrm{cpsd}}(A)$ admits a flat optimal solution, then $\xi_t^{\mathrm{cpsd}}(A) = \xi_{*}^{\mathrm{cpsd}}(A)$

$\xi_{\infty}^{\mathrm{cpsd}}(A)$ and $\xi_{*}^{\mathrm{cpsd}}(A)$

- We have $\xi_{t}^{\mathrm{cpsd}}(A) \to \xi_{\infty}^{\mathrm{cpsd}}(A)$, and if $\xi_{t}^{\mathrm{cpsd}}(A)$ admits a flat optimal solution, then $\xi_{t}^{\mathrm{cpsd}}(A) = \xi_{*}^{\mathrm{cpsd}}(A)$
- $\xi_{*}^{\mathrm{cpsd}}(A)$ is the minimum of $L(1)$ over all conic combinations L of trace evaluations at elements of the matrix positivity domain of $\{\sqrt{A_{ii}}x_i - x_i^2 : i \in [n]\}$ such that $A = (L(x_i x_j))$

$\xi_\infty^{\mathrm{cpsd}}(A)$ and $\xi_*^{\mathrm{cpsd}}(A)$

- We have $\xi_t^{\mathrm{cpsd}}(A) \to \xi_\infty^{\mathrm{cpsd}}(A)$, and if $\xi_t^{\mathrm{cpsd}}(A)$ admits a flat optimal solution, then $\xi_t^{\mathrm{cpsd}}(A) = \xi_*^{\mathrm{cpsd}}(A)$
- $\xi_*^{\mathrm{cpsd}}(A)$ is the minimum of $L(1)$ over all conic combinations L of trace evaluations at elements of the matrix positivity domain of $\{\sqrt{A_{ii}}x_i - x_i^2 : i \in [n]\}$ such that $A = (L(x_i x_j))$

$$\xi_*^{\mathrm{cpsd}}(A) = \inf\Big\{ \sum_{m=1}^{M} d_m \cdot \max_{i\in[n]} \frac{\|X_i^m\|^2}{A_{ii}} : M \in \mathbb{N},\ d_1, \ldots, d_M \in \mathbb{N},$$
$$X_i^m \in \mathcal{H}_+^{d_m} \text{ for } i \in [n], m \in [M],$$
$$A = \mathrm{Gram}\Big(\bigoplus_{m=1}^{M} X_1^m, \ldots, \bigoplus_{m=1}^{M} X_n^m \Big) \Big\}.$$

Lower bound [Prakash–Sikora–Varvitsiotis–Wei 2016]:

$$\frac{\left(\sum_{i=1}^{n} \sqrt{A_{ii}}\right)^2}{\sum_{i,j=1}^{n} A_{ij}} \leq \mathrm{cpsd\text{-}rank}(A)$$

Lower bound [Prakash–Sikora–Varvitsiotis–Wei 2016]:

$$\frac{\left(\sum_{i=1}^{n} \sqrt{A_{ii}}\right)^2}{\sum_{i,j=1}^{n} A_{ij}} \leq \mathrm{cpsd\text{-}rank}(A)$$

We have

$$\xi_1^{\mathrm{cpsd}}(A) \geq \frac{\left(\sum_{i=1}^{n} \sqrt{A_{ii}}\right)^2}{\sum_{i,j=1}^{n} A_{ij}}$$

Lower bound [Prakash–Sikora–Varvitsiotis–Wei 2016]:

$$\frac{\left(\sum_{i=1}^{n} \sqrt{A_{ii}}\right)^2}{\sum_{i,j=1}^{n} A_{ij}} \leq \mathrm{cpsd\text{-}rank}(A)$$

We have

$$\xi_1^{\mathrm{cpsd}}(A) \geq \frac{\left(\sum_{i=1}^{n} \sqrt{A_{ii}}\right)^2}{\sum_{i,j=1}^{n} A_{ij}}$$

Sharp for the matrix $A \in \mathbb{R}^{5\times 5}$ given by $A_{ij} = \cos\big(4\pi/5(i-j)\big)^2$

Extra constraints to go beyond $\xi_*^{\mathrm{cpsd}}(A)$

Let $X_1, \ldots, X_n$ be Hermitian PSD matrices s.t. $A_{ij} = \mathrm{Tr}(X_i X_j)$

Extra constraints to go beyond $\xi_*^{\mathrm{cpsd}}(A)$

Let $X_1, \ldots, X_n$ be Hermitian PSD matrices s.t. $A_{ij} = \mathrm{Tr}(X_i X_j)$

For each $v \in \mathbb{R}^n$, the following matrix is psd:

$$v^{\mathsf{T}} A v I - \Big(\sum_{i=1}^{n} v_i X_i \Big)^2$$

Extra constraints to go beyond $\xi_*^{\mathrm{cpsd}}(A)$

Let $X_1, \ldots, X_n$ be Hermitian PSD matrices s.t. $A_{ij} = \mathrm{Tr}(X_i X_j)$

For each $v \in \mathbb{R}^n$, the following matrix is psd:

$$v^{\mathsf{T}} A v I - \Big(\sum_{i=1}^{n} v_i X_i\Big)^2$$

We can use this to add additional constraints to $\xi_t^{\mathrm{cpsd}}(A)$ by extending the quadratic module

Extra constraints to go beyond $\xi_*^{\mathrm{cpsd}}(A)$

Let $X_1, \ldots, X_n$ be Hermitian PSD matrices s.t. $A_{ij} = \mathrm{Tr}(X_i X_j)$

For each $v \in \mathbb{R}^n$, the following matrix is psd:

$$v^{\mathsf{T}} A v I - \Big(\sum_{i=1}^{n} v_i X_i \Big)^2$$

We can use this to add additional constraints to $\xi_t^{\mathrm{cpsd}}(A)$ by extending the quadratic module

For a subset $V \subseteq S^{n-1}$ we have the stronger bound $\xi_{t,V}^{\mathrm{cpsd}}(A)$

Extra constraints to go beyond $\xi_*^{\mathrm{cpsd}}(A)$

Let $X_1, \ldots, X_n$ be Hermitian PSD matrices s.t. $A_{ij} = \mathrm{Tr}(X_i X_j)$

For each $v \in \mathbb{R}^n$, the following matrix is psd:

$$v^\mathsf{T} A v I - \Big(\sum_{i=1}^n v_i X_i \Big)^2$$

We can use this to add additional constraints to $\xi_t^{\mathrm{cpsd}}(A)$ by extending the quadratic module

For a subset $V \subseteq S^{n-1}$ we have the stronger bound $\xi_{t,V}^{\mathrm{cpsd}}(A)$

Example:

$$A = \begin{pmatrix} 1 & 1/2 & 0 & 0 & 1/2 \\ 1/2 & 1 & 1/2 & 0 & 0 \\ 0 & 1/2 & 1 & 1/2 & 0 \\ 0 & 0 & 1/2 & 1 & 1/2 \\ 1/2 & 0 & 0 & 1/2 & 1 \end{pmatrix}$$

$$\xi_1^{\mathrm{cpsd}}(A) = \xi_*^{\mathrm{cpsd}}(A) = \frac{5}{2}$$

Extra constraints to go beyond $\xi_*^{\mathrm{cpsd}}(A)$

Let $X_1, \ldots, X_n$ be Hermitian PSD matrices s.t. $A_{ij} = \mathrm{Tr}(X_i X_j)$

For each $v \in \mathbb{R}^n$, the following matrix is psd:

$$v^{\mathsf{T}} A v I - \Big(\sum_{i=1}^{n} v_i X_i \Big)^2$$

We can use this to add additional constraints to $\xi_t^{\mathrm{cpsd}}(A)$ by extending the quadratic module

For a subset $V \subseteq S^{n-1}$ we have the stronger bound $\xi_{t,V}^{\mathrm{cpsd}}(A)$

Example:

$$A = \begin{pmatrix} 1 & 1/2 & 0 & 0 & 1/2 \\ 1/2 & 1 & 1/2 & 0 & 0 \\ 0 & 1/2 & 1 & 1/2 & 0 \\ 0 & 0 & 1/2 & 1 & 1/2 \\ 1/2 & 0 & 0 & 1/2 & 1 \end{pmatrix}$$

$$\xi_1^{\mathrm{cpsd}}(A) = \xi_*^{\mathrm{cpsd}}(A) = \frac{5}{2},\ V = \Big\{ \frac{e_i + e_j}{\sqrt{2}} : i, j \in [5] \Big\}$$

Extra constraints to go beyond $\xi_*^{\mathrm{cpsd}}(A)$

Let $X_1, \ldots, X_n$ be Hermitian PSD matrices s.t. $A_{ij} = \mathrm{Tr}(X_i X_j)$

For each $v \in \mathbb{R}^n$, the following matrix is psd:

$$v^\mathsf{T} A v I - \Big(\sum_{i=1}^n v_i X_i\Big)^2$$

We can use this to add additional constraints to $\xi_t^{\mathrm{cpsd}}(A)$ by extending the quadratic module

For a subset $V \subseteq S^{n-1}$ we have the stronger bound $\xi_{t,V}^{\mathrm{cpsd}}(A)$

Example:

$$A = \begin{pmatrix} 1 & 1/2 & 0 & 0 & 1/2 \\ 1/2 & 1 & 1/2 & 0 & 0 \\ 0 & 1/2 & 1 & 1/2 & 0 \\ 0 & 0 & 1/2 & 1 & 1/2 \\ 1/2 & 0 & 0 & 1/2 & 1 \end{pmatrix}$$

$\xi_1^{\mathrm{cpsd}}(A) = \xi_*^{\mathrm{cpsd}}(A) = \dfrac{5}{2},\ V = \left\{ \dfrac{e_i + e_j}{\sqrt{2}} : i,j \in [5] \right\},\ \xi_{2,V}^{\mathrm{cpsd}}(A) = \dfrac{10}{3}$

The completely positive rank (cp-rank)

Fawzi and Parrilo (2014) give this SDP to lower bound $\mathrm{cp\text{-}rank}(A)$:

$$\begin{aligned}
\tau_{\mathrm{cp}}^{\mathrm{sos}}(A) = \inf\Big\{ & \alpha : \alpha \in \mathbb{R},\, X \in \mathbb{R}^{n^2 \times n^2}, \\
& \begin{pmatrix} \alpha & \mathrm{vec}(A)^\mathsf{T} \\ \mathrm{vec}(A) & X \end{pmatrix} \succeq 0, \\
& X_{(i,j),(i,j)} \leq A_{ij}^2 \quad \text{for} \quad 1 \leq i, j \leq n, \\
& X_{(i,j),(k,l)} = X_{(i,l),(k,j)} \quad \text{for} \quad 1 \leq i < k \leq n,\ 1 \leq j < l \leq n, \\
& X \preceq A \otimes A \Big\}.
\end{aligned}$$

The completely positive rank (cp-rank)

Fawzi and Parrilo (2014) give this SDP to lower bound $\mathrm{cp\text{-}rank}(A)$:

$$
\begin{aligned}
\tau_{\mathrm{cp}}^{\mathrm{sos}}(A) = \inf\Big\{ & \alpha : \alpha \in \mathbb{R},\, X \in \mathbb{R}^{n^2 \times n^2}, \\
& \begin{pmatrix} \alpha & \mathrm{vec}(A)^\mathsf{T} \\ \mathrm{vec}(A) & X \end{pmatrix} \succeq 0, \\
& X_{(i,j),(i,j)} \leq A_{ij}^2 \quad \text{for} \quad 1 \leq i, j \leq n, \\
& X_{(i,j),(k,l)} = X_{(i,l),(k,j)} \quad \text{for} \quad 1 \leq i < k \leq n,\ 1 \leq j < l \leq n, \\
& X \preceq A \otimes A \Big\}.
\end{aligned}
$$

They derive $\tau_{\mathrm{cp}}^{\mathrm{sos}}(A)$ as an SDP relaxation of

$$
\tau_{\mathrm{cp}}(A) = \min \left\{ \alpha : \alpha > 0,\ \frac{1}{\alpha} A \in \mathrm{conv}\big\{ R \in \mathcal{S}^n : 0 \leq R \leq A,\ R \preceq A,\ \mathrm{rank}(R) \leq 1 \big\} \right\}
$$

The completely positive rank (cp-rank)

Fawzi and Parrilo (2014) give this SDP to lower bound $\mathrm{cp\text{-}rank}(A)$:

$$\begin{aligned}\tau_{\mathrm{cp}}^{\mathrm{sos}}(A) = \inf\Big\{\alpha : \alpha \in \mathbb{R},\, X \in \mathbb{R}^{n^2\times n^2},\\ &\begin{pmatrix}\alpha & \mathrm{vec}(A)^{\mathsf{T}}\\ \mathrm{vec}(A) & X\end{pmatrix} \succeq 0,\\ &X_{(i,j),(i,j)} \le A_{ij}^2 \quad \text{for} \quad 1 \le i,j \le n,\\ &X_{(i,j),(k,l)} = X_{(i,l),(k,j)} \quad \text{for} \quad 1 \le i < k \le n,\ 1 \le j < l \le n,\\ &X \preceq A \otimes A\Big\}.\end{aligned}$$

They derive $\tau_{\mathrm{cp}}^{\mathrm{sos}}(A)$ as an SDP relaxation of

$$\tau_{\mathrm{cp}}(A) = \min\Big\{\alpha : \alpha > 0,\, \frac{1}{\alpha}A \in \mathrm{conv}\big\{R \in \mathcal{S}^n : 0 \le R \le A,\, R \preceq A,\, \mathrm{rank}(R) \le 1\big\}\Big\}$$

$\tau_{\mathrm{cp}}(A)$ is at least the rank of A and the fractional edge-clique cover number of the support graph of A

Adapting our hierarchy for the $\mathrm{cp\text{-}rank}$

Suppose $A_{ij} = v_i^\mathsf{T} v_j$ for $v_1, \ldots, v_n \in \mathbb{R}_+^d$

Adapting our hierarchy for the cp-rank

Suppose $A_{ij} = v_i^{\mathsf{T}} v_j$ for $v_1, \ldots, v_n \in \mathbb{R}_+^d$

Then, $A_{ij} = \mathrm{Tr}(X_i X_j)$ for diagonal PSD matrices $X_i = \mathrm{Diag}(v_i)$

Adapting our hierarchy for the $\mathrm{cp\text{-}rank}$

Suppose $A_{ij} = v_i^\mathsf{T} v_j$ for $v_1, \ldots, v_n \in \mathbb{R}_+^d$

Then, $A_{ij} = \mathrm{Tr}(X_i X_j)$ for diagonal PSD matrices $X_i = \mathrm{Diag}(v_i)$

Use ideas for cpsd-rank to derive a hierarchy for cp-rank

Adapting our hierarchy for the $\mathrm{cp\text{-}rank}$

Suppose $A_{ij} = v_i^\mathsf{T} v_j$ for $v_1, \ldots, v_n \in \mathbb{R}_+^d$

Then, $A_{ij} = \mathrm{Tr}(X_i X_j)$ for diagonal PSD matrices $X_i = \mathrm{Diag}(v_i)$

Use ideas for cpsd-rank to derive a hierarchy for cp-rank

$$\mathcal{M}_{2t}(S) = \mathrm{cone}\{gp^2 : g \in S \cup \{1\},\ p \in \mathbb{R}[\mathbf{x}],\ \deg(gp^2) \leq 2t\}$$

Adapting our hierarchy for the cp-rank

Suppose $A_{ij} = v_i^\mathsf{T} v_j$ for $v_1, \ldots, v_n \in \mathbb{R}_+^d$

Then, $A_{ij} = \mathrm{Tr}(X_i X_j)$ for diagonal PSD matrices $X_i = \mathrm{Diag}(v_i)$

Use ideas for cpsd-rank to derive a hierarchy for cp-rank

$$\mathcal{M}_{2t}(S) = \mathrm{cone}\{gp^2 : g \in S \cup \{1\},\, p \in \mathbb{R}[\mathbf{x}],\, \deg(gp^2) \leq 2t\}$$

$$S = \{\sqrt{A_{ii}} x_i - x_i^2\} \cup \{A_{ij} - x_i x_j : 1 \leq i < j \leq n\}$$

Adapting our hierarchy for the cp-rank

Suppose $A_{ij} = v_i^\mathsf{T} v_j$ for $v_1, \ldots, v_n \in \mathbb{R}^d_+$

Then, $A_{ij} = \mathrm{Tr}(X_i X_j)$ for diagonal PSD matrices $X_i = \mathrm{Diag}(v_i)$

Use ideas for cpsd-rank to derive a hierarchy for cp-rank

$$\mathcal{M}_{2t}(S) = \mathrm{cone}\{gp^2 : g \in S \cup \{1\},\ p \in \mathbb{R}[\mathbf{x}],\ \deg(gp^2) \leq 2t\}$$

$$S = \{\sqrt{A_{ii}}x_i - x_i^2\} \cup \{A_{ij} - x_i x_j : 1 \leq i < j \leq n\}$$

$$\xi_t^{\mathrm{cp}}(A) = \min\Big\{L(1) : L \in \mathbb{R}[x_1, \ldots, x_n]^*_{2t},\ (L(x_i x_j)) = A,\ L \geq 0 \quad \text{on} \quad \mathcal{M}_{2t}(S)\Big\}$$

Adapting our hierarchy for the cp-rank

Suppose $A_{ij} = v_i^\mathsf{T} v_j$ for $v_1, \ldots, v_n \in \mathbb{R}^d_+$

Then, $A_{ij} = \mathrm{Tr}(X_i X_j)$ for diagonal PSD matrices $X_i = \mathrm{Diag}(v_i)$

Use ideas for cpsd-rank to derive a hierarchy for cp-rank

$$\mathcal{M}_{2t}(S) = \mathrm{cone}\{gp^2 : g \in S \cup \{1\},\, p \in \mathbb{R}[\mathbf{x}],\, \deg(gp^2) \leq 2t\}$$

$$S = \{\sqrt{A_{ii}}x_i - x_i^2\} \cup \{A_{ij} - x_i x_j : 1 \leq i < j \leq n\}$$

$$\xi_t^{\mathrm{cp}}(A) = \min\Big\{L(1) : L \in \mathbb{R}[x_1, \ldots, x_n]^*_{2t},\; (L(x_i x_j)) = A,\; L \geq 0 \quad \text{on} \quad \mathcal{M}_{2t}(S)\Big\}$$

$$\xi_1^{\mathrm{cp}}(A) \leq \ldots \leq \xi_\infty^{\mathrm{cp}}(A) = \xi_*^{\mathrm{cp}}(A) \leq \mathrm{cp}\text{-rank}(A)$$

Extra constraints for the $\mathrm{cp\text{-}rank}$

As in the $\mathrm{cpsd\text{-}rank}$ case we can add extra constraints for a set $V \subseteq S^{n-1}$ giving the stronger bound $\xi_{t,V}^{\mathrm{cp}}(A)$

Extra constraints for the $\mathrm{cp\text{-}rank}$

As in the $\mathrm{cpsd\text{-}rank}$ case we can add extra constraints for a set $V \subseteq S^{n-1}$ giving the stronger bound $\xi_{t,V}^{\mathrm{cp}}(A)$

We have $\xi_{*,S^{n-1}}^{\mathrm{cp}}(A) = \tau_{\mathrm{cp}}(A)$

Extra constraints for the cp-rank

As in the cpsd-rank case we can add extra constraints for a set $V \subseteq S^{n-1}$ giving the stronger bound $\xi^{\mathrm{cp}}_{t,V}(A)$

We have $\xi^{\mathrm{cp}}_{*,S^{n-1}}(A) = \tau_{\mathrm{cp}}(A)$

Let $V_1 \subseteq V_2 \subseteq \ldots \subseteq S^{n-1}$ be finite subsets such that $\bigcup_k V_k$ is dense in S^{n-1}

We have $\xi^{\mathrm{cp}}_{*,V_k}(A) \to \xi^{\mathrm{cp}}_{*,S^{n-1}}(A)$ as $k \to \infty$

Extra constraints for the $\mathrm{cp\text{-}rank}$

As in the $\mathrm{cpsd\text{-}rank}$ case we can add extra constraints for a set $V \subseteq S^{n-1}$ giving the stronger bound $\xi^{\mathrm{cp}}_{t,V}(A)$

We have $\xi^{\mathrm{cp}}_{*,S^{n-1}}(A) = \tau_{\mathrm{cp}}(A)$

Let $V_1 \subseteq V_2 \subseteq \ldots \subseteq S^{n-1}$ be finite subsets such that $\bigcup_k V_k$ is dense in S^{n-1}

We have $\xi^{\mathrm{cp}}_{*,V_k}(A) \to \xi^{\mathrm{cp}}_{*,S^{n-1}}(A)$ as $k \to \infty$

This gives a (doubly indexed) sequence of finite semidefinite programs converging asymptotically to $\tau_{\mathrm{cp}}(A)$

More efficient tensor constraints

Let $\xi_{t,+}^{\mathrm{cp}}(A)$ be the following strengthening of $\xi_t^{\mathrm{cp}}(A)$:

More efficient tensor constraints

Let $\xi_{t,+}^{\mathrm{cp}}(A)$ be the following strengthening of $\xi_t^{\mathrm{cp}}(A)$:

- Add entrywise nonnegativity constraints

More efficient tensor constraints

Let $\xi_{t,+}^{\mathrm{cp}}(A)$ be the following strengthening of $\xi_t^{\mathrm{cp}}(A)$:

- Add entrywise nonnegativity constraints
- Add the tensor constraint $X \preceq A \otimes A$ from $\tau_{\mathrm{cp}}^{\mathrm{sos}}(A)$:

$$(L(ww'))_{w,w' \in \langle \mathbf{x} \rangle_{=l}} \preceq A^{\otimes l} \quad \text{for} \quad 2 \leq l \leq t$$

More efficient tensor constraints

Let $\xi_{t,+}^{\mathrm{cp}}(A)$ be the following strengthening of $\xi_t^{\mathrm{cp}}(A)$:

- Add entrywise nonnegativity constraints
- Add the tensor constraint $X \preceq A \otimes A$ from $\tau_{\mathrm{cp}}^{\mathrm{sos}}(A)$:

$$(L(ww'))_{w,w' \in \langle \mathbf{x} \rangle_{=l}} \preceq A^{\otimes l} \quad \text{for} \quad 2 \leq l \leq t$$

- Implement this constraint more efficiently by exploiting symmetry:

$$(L(mm'))_{m,m' \in [\mathbf{x}]_{=l}} \preceq Q_l A^{\otimes l} Q_l^{\mathsf{T}} \quad \text{for} \quad 2 \leq l \leq t$$

More efficient tensor constraints

Let $\xi_{t,+}^{\mathrm{cp}}(A)$ be the following strengthening of $\xi_t^{\mathrm{cp}}(A)$:

- Add entrywise nonnegativity constraints
- Add the tensor constraint $X \preceq A \otimes A$ from $\tau_{\mathrm{cp}}^{\mathrm{sos}}(A)$:

$$(L(ww'))_{w,w' \in \langle \mathbf{x} \rangle_{=l}} \preceq A^{\otimes l} \quad \text{for} \quad 2 \leq l \leq t$$

- Implement this constraint more efficiently by exploiting symmetry:

$$(L(mm'))_{m,m' \in [\mathbf{x}]_{=l}} \preceq Q_l A^{\otimes l} Q_l^{\mathsf{T}} \quad \text{for} \quad 2 \leq l \leq t$$

Then $\xi_{2,+}^{\mathrm{cp}}(A)$ is a more efficient strengthening of $\tau_{\mathrm{cp}}^{\mathrm{sos}}(A)$

The nonnegative rank

The nonnegative rank $\mathrm{rank}_+(A)$ is the smallest d for which there are vectors $u_1, \ldots, u_n, v_1, \ldots, v_n \in \mathbb{R}_+^d$ such that $A_{ij} = u_i^\mathsf{T} v_j$

The nonnegative rank

The nonnegative rank $\mathrm{rank}_+(A)$ is the smallest d for which there are vectors $u_1, \ldots, u_n, v_1, \ldots, v_n \in \mathbb{R}_+^d$ such that $A_{ij} = u_i^\mathsf{T} v_j$

Relevant for the extension complexity of linear programs

The nonnegative rank

The nonnegative rank $\mathrm{rank}_+(A)$ is the smallest d for which there are vectors $u_1, \ldots, u_n, v_1, \ldots, v_n \in \mathbb{R}^d_+$ such that $A_{ij} = u_i^\mathsf{T} v_j$

Relevant for the extension complexity of linear programs

Fawzi and Parrilo (2014) define relaxations
$\tau_+^{\mathrm{sos}}(A) \leq \tau_+(A) \leq \mathrm{rank}_+(A)$

The nonnegative rank

The nonnegative rank $\mathrm{rank}_+(A)$ is the smallest d for which there are vectors $u_1, \ldots, u_n, v_1, \ldots, v_n \in \mathbb{R}_+^d$ such that $A_{ij} = u_i^{\mathsf{T}} v_j$

Relevant for the extension complexity of linear programs

Fawzi and Parrilo (2014) define relaxations $\tau_+^{\mathrm{sos}}(A) \leq \tau_+(A) \leq \mathrm{rank}_+(A)$

For $A \in \mathbb{R}_+^{m \times n}$ there are positive semidefinite diagonal matrices $X_1, \ldots, X_{m+n}$ with $A_{ij} = \mathrm{Tr}(X_i X_{m+j})$ and $\lambda_{\max}(X_i)^2 \leq \max_{i,j} A_{ij}$

The nonnegative rank

The nonnegative rank $\mathrm{rank}_+(A)$ is the smallest d for which there are vectors $u_1, \ldots, u_n, v_1, \ldots, v_n \in \mathbb{R}_+^d$ such that $A_{ij} = u_i^\mathsf{T} v_j$

Relevant for the extension complexity of linear programs

Fawzi and Parrilo (2014) define relaxations $\tau_+^{\mathrm{sos}}(A) \leq \tau_+(A) \leq \mathrm{rank}_+(A)$

For $A \in \mathbb{R}_+^{m \times n}$ there are positive semidefinite diagonal matrices $X_1, \ldots, X_{m+n}$ with $A_{ij} = \mathrm{Tr}(X_i X_{m+j})$ and $\lambda_{\max}(X_i)^2 \leq \max_{i,j} A_{ij}$

We can use this to adapt the above techniques to give a hiearchy

$$\xi_1^+(A) \leq \ldots \leq \xi_\infty^+(A) = \xi_*^+(A) = \tau_+(A) \leq \mathrm{rank}_+(A).$$

The nonnegative rank

The nonnegative rank $\mathrm{rank}_+(A)$ is the smallest d for which there are vectors $u_1, \ldots, u_n, v_1, \ldots, v_n \in \mathbb{R}_+^d$ such that $A_{ij} = u_i^\mathsf{T} v_j$

Relevant for the extension complexity of linear programs

Fawzi and Parrilo (2014) define relaxations $\tau_+^{\mathrm{sos}}(A) \leq \tau_+(A) \leq \mathrm{rank}_+(A)$

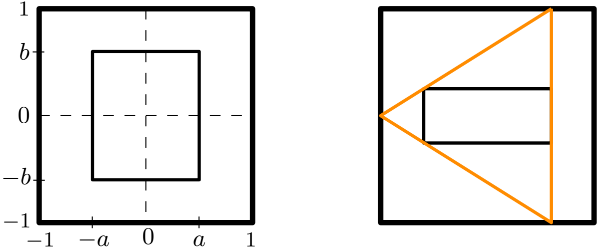
For $A \in \mathbb{R}_+^{m \times n}$ there are positive semidefinite diagonal matrices $X_1, \ldots, X_{m+n}$ with $A_{ij} = \mathrm{Tr}(X_i X_{m+j})$ and $\lambda_{\max}(X_i)^2 \leq \max_{i,j} A_{ij}$

We can use this to adapt the above techniques to give a hiearchy

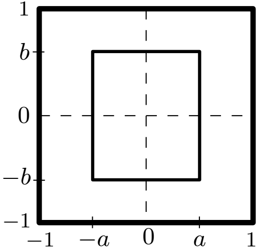
$$\xi_1^+(A) \leq \ldots \leq \xi_\infty^+(A) = \xi_*^+(A) = \tau_+(A) \leq \mathrm{rank}_+(A).$$

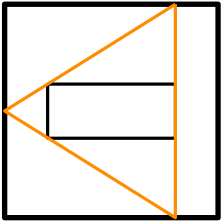
Going back to tracial optimization we can adapt this to the psd-rank – still work in progress

Nested rectangle problem [Fawzi–Parrilo, 2016]:



Nested rectangle problem [Fawzi–Parrilo, 2016]:

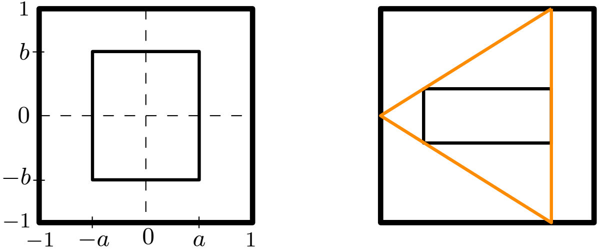




Such a triangle exists if and only if

$$\operatorname{rank}_+\left(\begin{pmatrix} 1-a & 1+a & 1+a & 1-a \\ 1+a & 1-a & 1-a & 1+a \\ 1-b & 1-b & 1+b & 1+b \\ 1+b & 1+b & 1-b & 1-b \end{pmatrix} \right) \leq 3$$

Nested rectangle problem [Fawzi–Parrilo, 2016]:

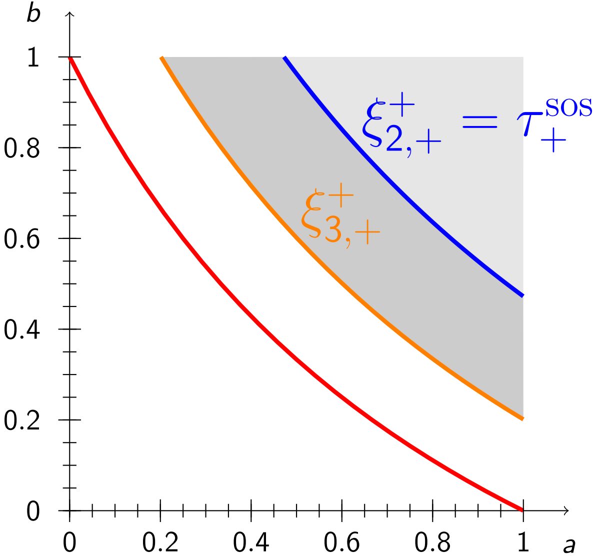


Such a triangle exists if and only if

$$\mathrm{rank}_+\left(\begin{pmatrix} 1-a & 1+a & 1+a & 1-a \\ 1+a & 1-a & 1-a & 1+a \\ 1-b & 1-b & 1+b & 1+b \\ 1+b & 1+b & 1-b & 1-b \end{pmatrix} \right) \leq 3$$

In fact, such a triangle exists if and only if $(1+a)(1+b) \leq 2$

Nested rectangle problem [Fawzi–Parrilo, 2016]:



Thank you!